Cognitive Science Tutorial II:

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Empirical Investigations of Mathematical Skills

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Overview

- Basics:
  - Core systems of number
  - Representational effects
- Case studies:

Topic	Method(s)	Reference(s)
Arithmetic: decimal effects	Experimentation	Neth (2004)
Arithmetic: notational properties	Experimentation	Landy & Goldstone (2007)
Arithmetic: inversion problems	Experimentation, microgenetic analysis	Siegler & Stern (1998)
Geometry: expert proofs	Computational modeling, expertise research	Koedinger & Anderson (1990)
Algebra: equation solving	Computational modeling, fMRI	Anderson (2005, 2007)

Mathematical Cognition

What is a number, that a man may know it,
and a man, that he may know a number?

Warren McCulloch (1965)

- A young, but booming discipline
- Motivations:
  - experimental psychology: well-defined domain, simple correlates of mental process
  - brain mapping: neural correlates
  - learning & instruction: math as key skill

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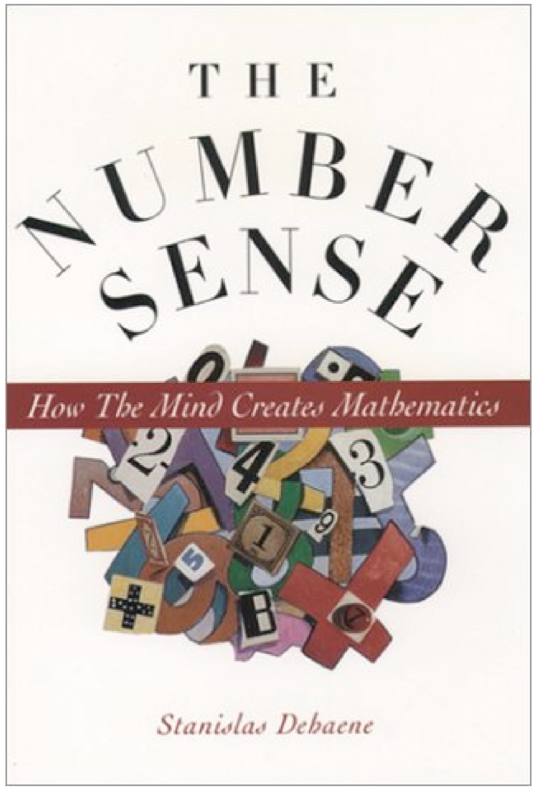
Issues & Recommendation

- Multiple levels:
  - human adults ⇄ infants ⇄ animals
  - mathematical constructs ⇄ mental representations
    ⇄ brain implementation
  - continuous magnitude/quantity ⇄ discrete symbols
    ⇄ verbal labels

- Recommended reading:
  Stanislas Dehaene (1997):
  The Number Sense: How the mind creates mathematics. OUP.



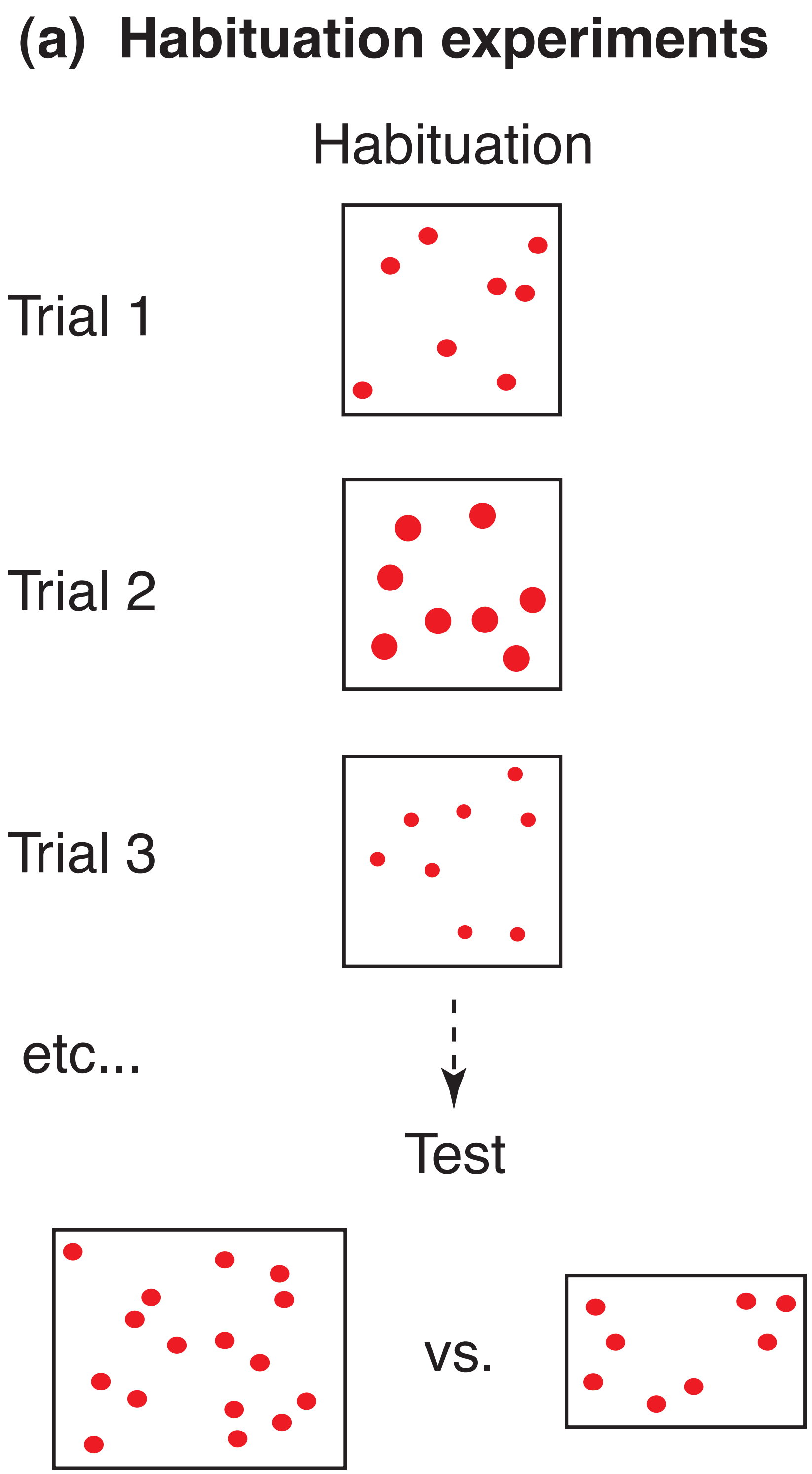
Biological Basis & Constraints

- Evolution of number sense
  - phylogenetic: numerosity in other species
  - ontogenetic: infants → adults
- Sources of evidence:
  - Homologies, e.g., distance and number size effects
  - Lesion studies
  - Brain circuitry
- 2 core systems of number (Feigenson, Dehaene, Spelke, 2004)

System 1: Approximate Magnitude

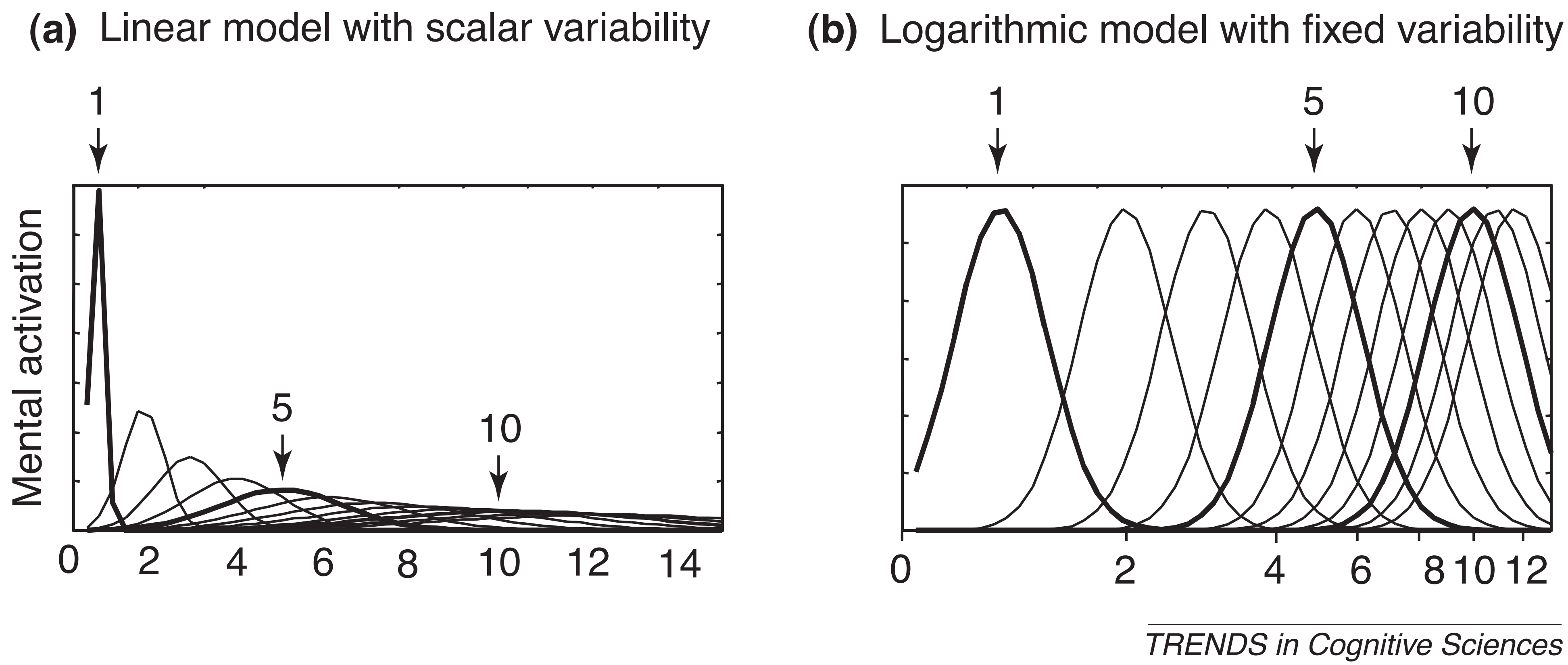
- Example: Xu & Spelke (2000)

- Fuzzy representation of magnitude ("how much")
  - ratio limits: 8:16 > 6:9 > 7:8
  - fails for small numerosities (1 vs. 2, 2 vs. 4, 2 vs. 3)
  - multimodal abstraction



System 1: Representation

- Numerosity as a fluctuating mental magnitude, measured on a continuous number-line
  - 2 alternative mathematical models:



  - increasing overlap for larger numerosities

System II: Precise Numerosity

- Feigenson et al. (2002):

- Keeping track of small numbers of individual objects ("how many")
  - upper bound of 3: 2 vs. 3, but not 3 vs. 4, 2 vs. 4
  - confused by continuous quantity (if 1 larger than 2)
  - multimodal abstraction

(c) Cracker choice experiments

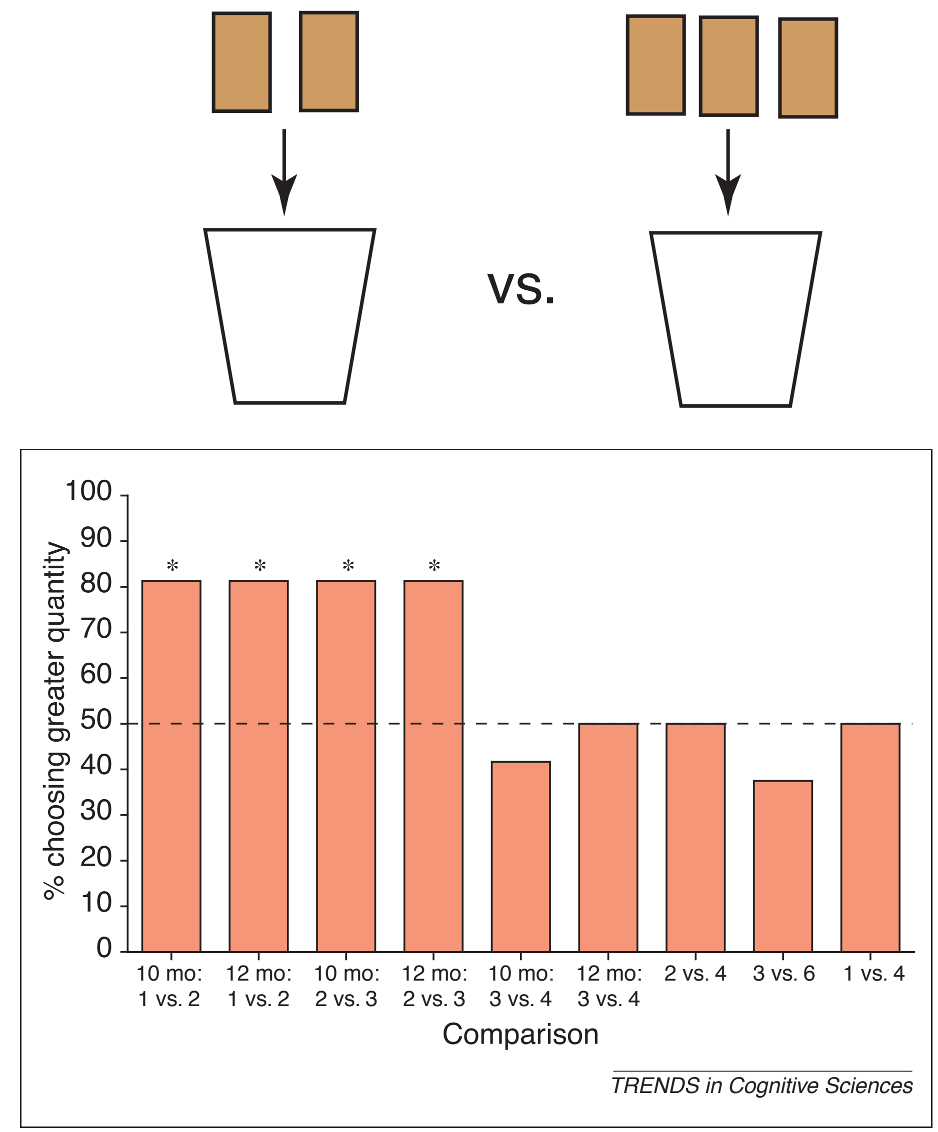


Figure 3. Infants' choices in the experiment by Feigenson *et al.* [20]. Bars represent the percentage of infants in each comparison group (at two different ages, 10 and 12 months, for the smaller quantities) choosing the greater quantity of crackers. Infants' choices demonstrate the set-size signature of the system for representing small numbers of numerically distinct individuals (Core system 2), in that infants performed randomly (dotted line at 50%) when either array contained more than 3 objects, even with highly discriminable ratios between the quantities. Asterisks denote significance levels of $P < 0.05$. Adapted with permission from [20].

Non-humans & Cerebral Correlates

- Shared heritage:
  - pigeons
  - rats
  - rhesus-monkeys
- Cerebral basis:
  - system 1: IPS
    - lesions
    - single-cell recordings
    - brain imaging
  - system 2: ???

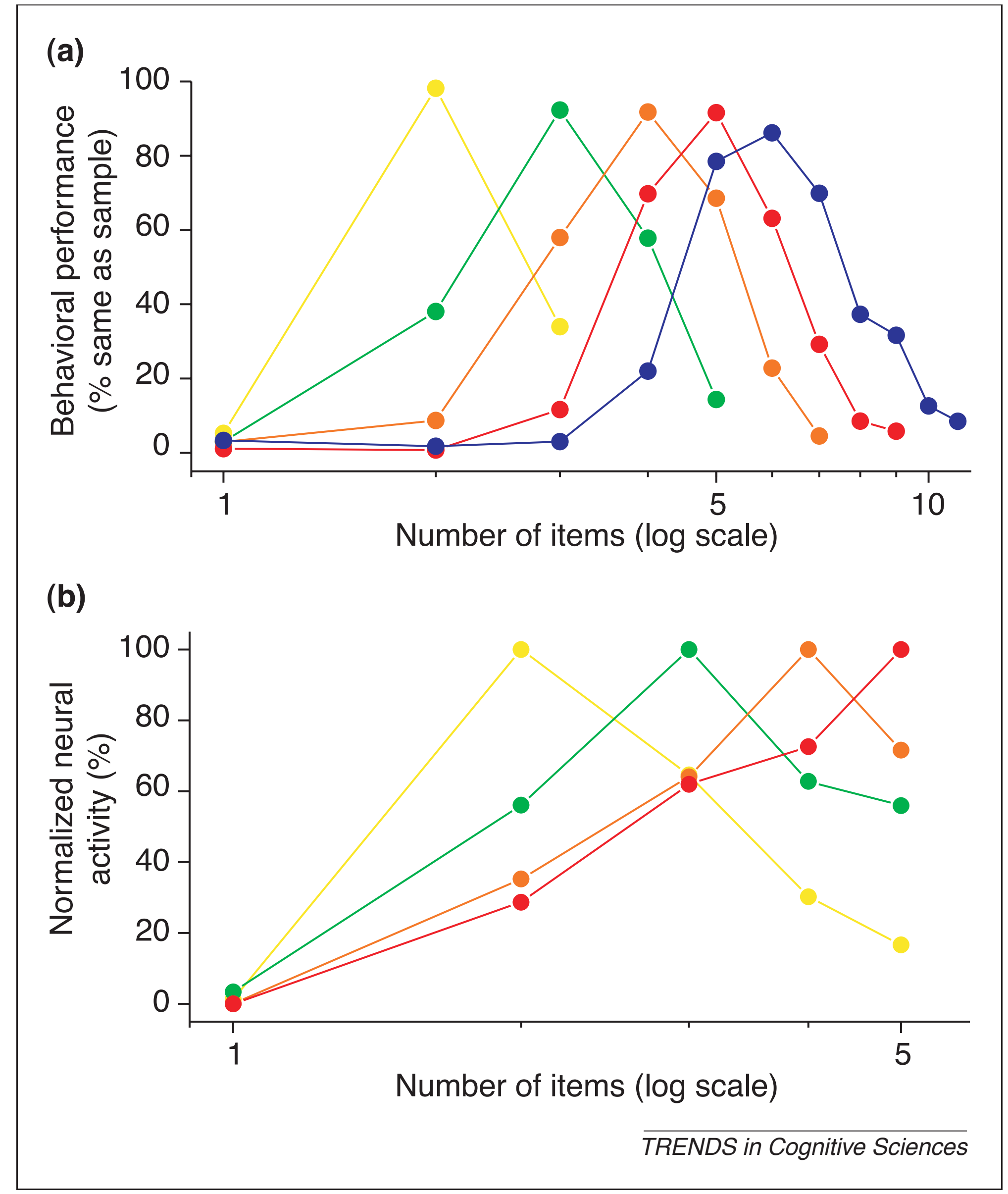


Figure 5. Behavioral and neural numerical filter functions. **(a)** The behavioral performance for two monkeys indicated whether they judged the first test stimulus (in a delayed match-to-numerosity task) as containing the same number of items as the sample display. The function peaks indicate the sample numerosity at which each curve was derived. Behavioral filter functions are plotted on a logarithmic scale. **(b)** Single-neuron representation of different numerosities in the prefrontal cortex of the same behaving monkeys. Population neural filter functions were derived by averaging the normalized single-unit activity for all neurons that preferred a given numerosity and transforming them to a logarithmic scale. Reprinted with permission from [52].

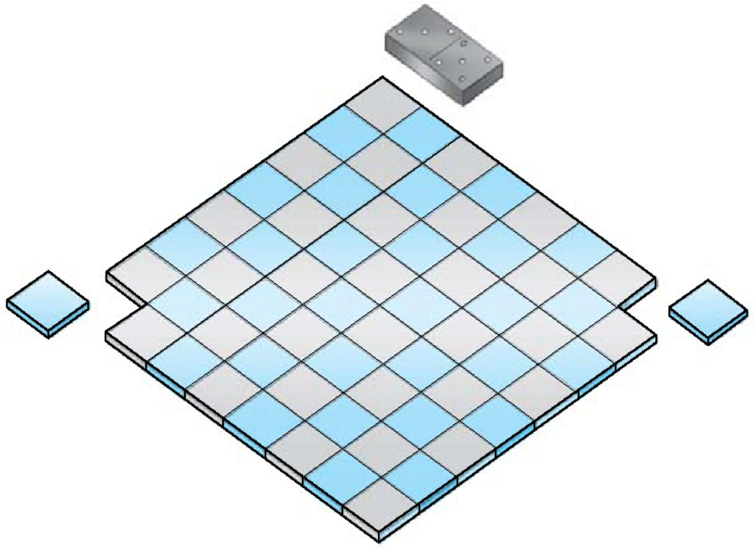
Other "Evolutions"

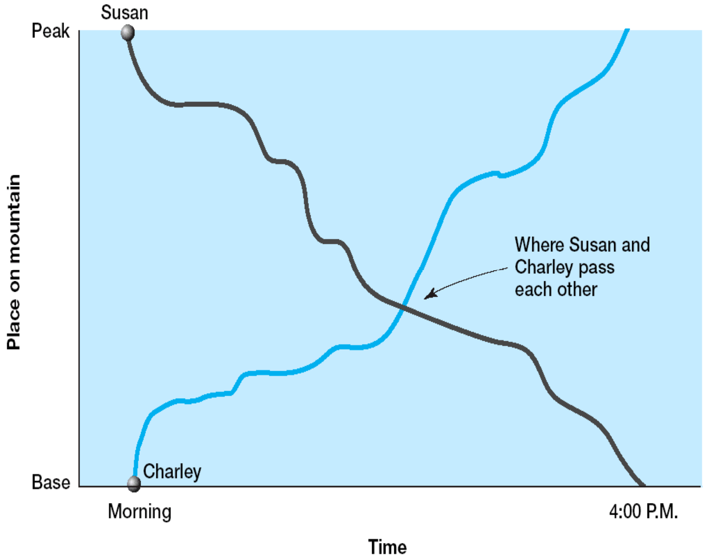
- Cultural evolution, mediated by social interaction, language, writing, etc.
- Discovery & design of (mathematical) artifacts:
  - Notations: symbols, number systems, formalisms
  - Tools: logarithms, calculating devices
- Intra-disciplinary evolution of mathematical ideas and formalisms
- Importance of the "right representations"...

Representational Effects

The solution to a problem changes the problem.
Peer's Law

- "Solving a problem simply means representing it so as to make the solution transparent." (Simon, 1996, p. 132)
- Are all deductive derivations 'merely' changes in representation?
- Example: Insight problems (e.g. mutilated chessboard, monk-mountain-problem...)



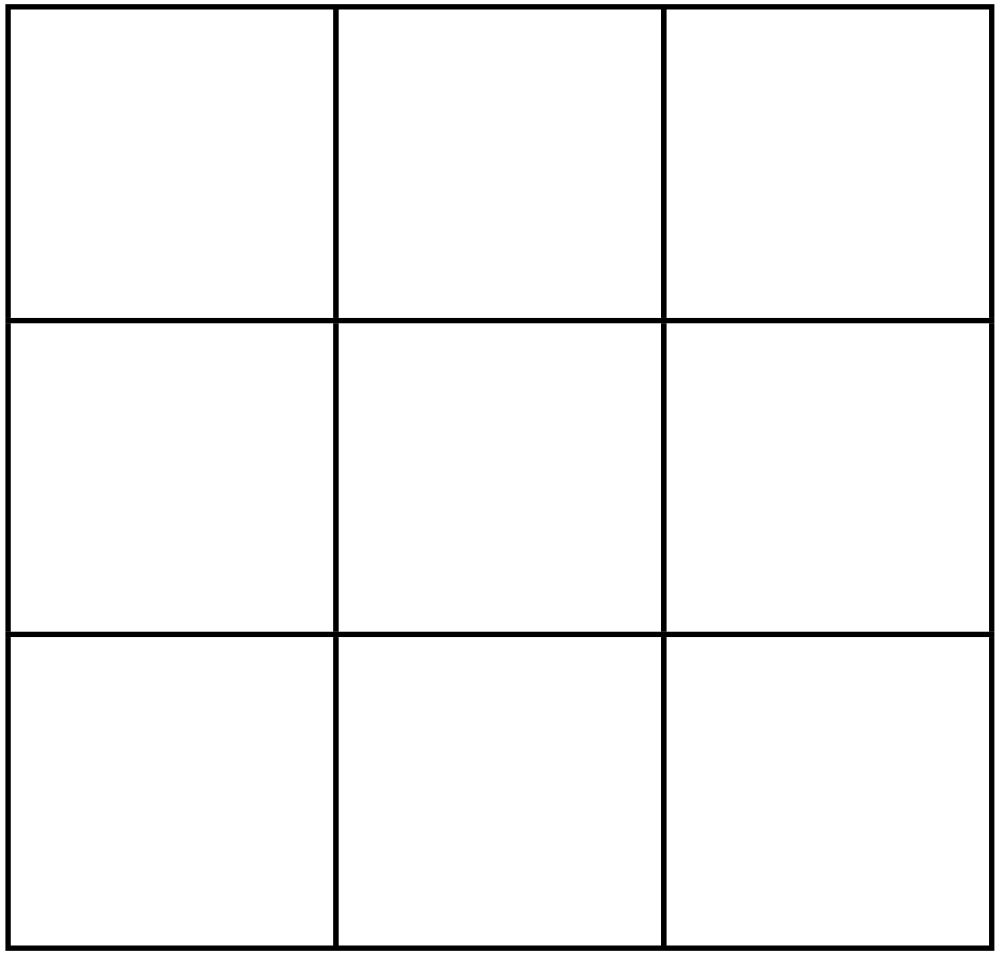


Demo: Number Scrabble

- Rules:
  - $\exists(1, 2, 3 \ldots 9)$
  - 2 players alternate draws (w/o replacement)
  - Goal: Get 3 numbers that sum to 15 (asap)
- Game:
  - available: 1 2 3 4 5 6 7 8 9
  - Player A:
  - Player B:

Demo: Tic-Tac-Toe

- 3x3 grid:



- Two players (X vs. O) alternate moves
- Goal: Select 3 in-a-row

Problem Isomorphs

- Number Scrabble ⇄ Tic Tac Toe

4	9	2
3	5	7
8	1	6

- same problem space (state space, operators, start & goal states)
- informational equivalence, but computational differences (Simon, 1978; Larkin & Simon, 1987)

Yet More Isomorphs...

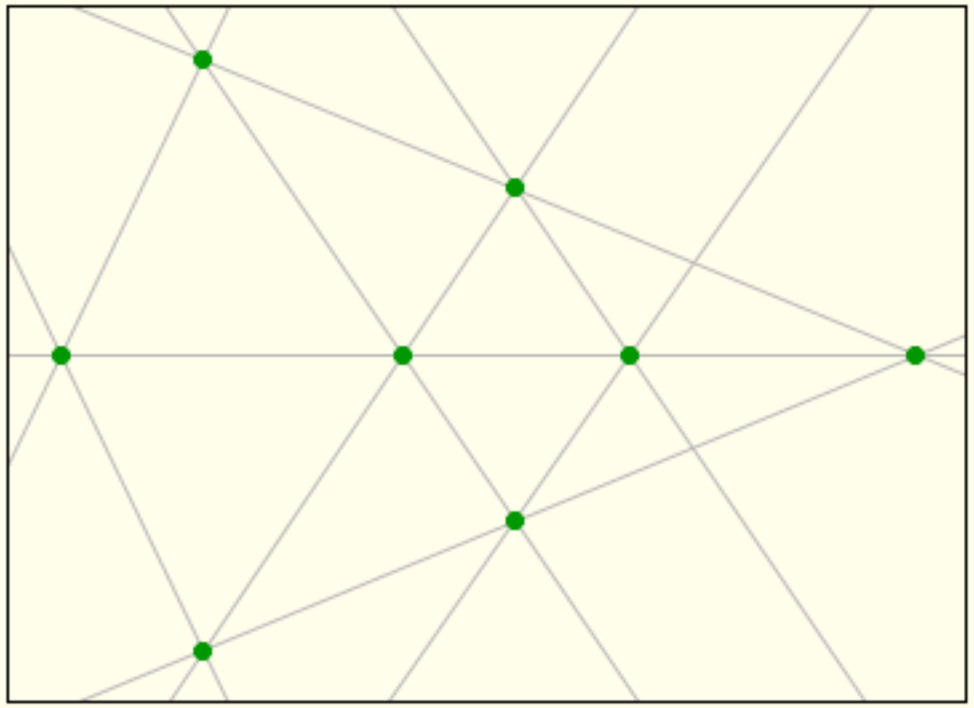
- Fish-Soup:
  - (fish, soup, swan, girl, horn, army, knit, vote, chat)
  - Goal: Select 3 words that share a letter

fish	soup	swan
girl	horn	army
knit	vote	chat

  - Play at www.cut-the-knot.org/SimpleGames/SoupFish.shtml

Yet More Isomorphs...

- The game of JAM (Michon, 1967):
  - Network of 9 roads and 8 cities



  - Goal: Take all roads that pass through a city
  - Play at www.cut-the-knot.org/SimpleGames/Jam.shtml

Case Study 1: Decimal Effects

- Neth & Payne (2001), Neth (2004): 'Thinking by Doing'
- Task domain of mental arithmetic:
  - easy to manipulate, well-defined and -researched
  - traditionally 'Platonic' realm, to be done 'in-the-head'
- Hypothesis: Not entirely 'in the head'
  - Notational effects
  - Effects of tools & "digital manipulations"

"Environmental arithmetic"

Mind Mechanics?

- Babbage's Conundrum:
  The most important part of the Analytical Engine was undoubtedly the mechanical method of carrying the tens. On this I laboured incessantly, each succeeding improvement advancing me a step or two. (…) At last I came to the conclusion that (…) nothing but teaching the Engine to foresee and then to act upon that foresight could ever lead me to the object I desired…

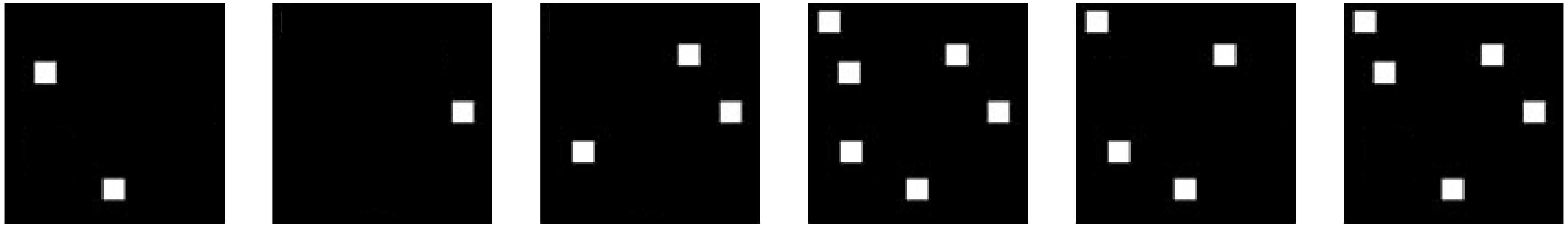
  Charles S. Babbage (1864), Passages from the Life of a Philosopher, Ch. VIII

- Background Phenomena

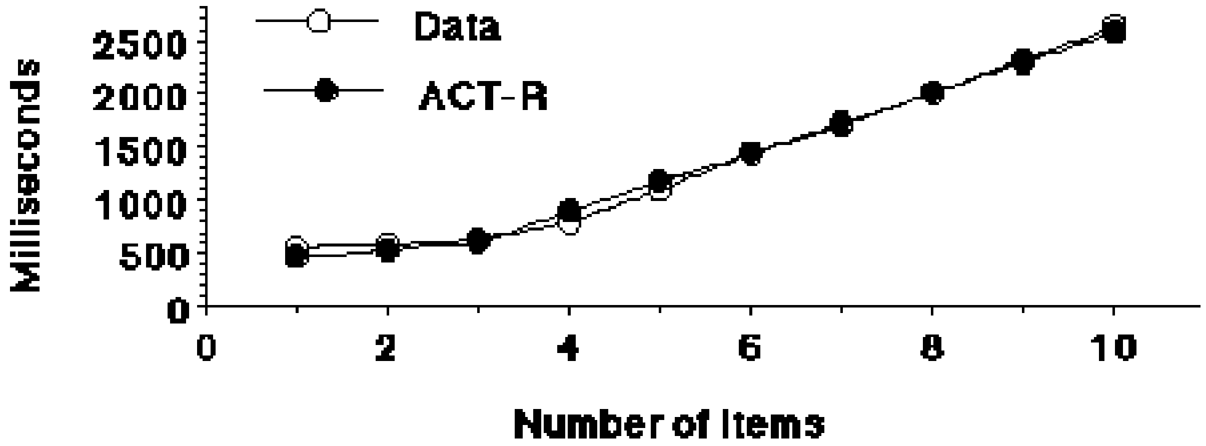
1. Mental representation of number line
2. Notation of numerals: 'four', IIII, IV, 4, ...
3. Arithmetic strategies: production vs. memory retrieval
4. Problem size effect

Subitizing (Core System II)

- Demo: 1, 2, 3 vs. many



- Typical recognition latencies:



The Mental Number Line

- Number comparison task (e.g., Moyer & Landauer, 1967):
  - 3 vs. 9 easier than 3 vs. 6 (distance effect)
  - 13 vs. 19 easier than 43 vs. 49 (magnitude effect)

- Effects hold across species (rats, pigeons, humans...)

=> analogical quantity representation with increasing fuzziness

Notations & Number Systems

Table 1.1: Different representational systems to represent numbers.

	one	two	three	four	five	six	seven	eight	nine	ten
(a) Tally system:	I	II	III	IIII	~~IIII~~	~~IIII~~ I	~~IIII~~ II	~~IIII~~ III	~~IIII~~ IIII	~~IIII~~ ~~IIII~~
(b) Greek letters:	α	β	γ	δ	ϵ	ς	ζ	η	θ	ι
(c) Roman numerals:	I	II	III	IV	V	VI	VII	VIII	IX	X
(d) Arabic decimal:	1	2	3	4	5	6	7	8	9	10
(e) Binary digits:	1	10	11	100	101	110	111	1000	1001	1010

- Different notations result in various trade-offs:
  - odd or even: binary > arabic > tally
  - addition: roman > arabic
  - multiplication & division: arabic > roman
- Note: informational, but no computational equivalence

Methods & Strategies

- Three plus five? (Groen & Parkman, 1972; Siegler & Shrager, 1984)
  - (1+1+1) + (1+1+1+1+1) → (1+1+1+1+1+1+1+1)
  - 3 + (1+1+1+1+1)
  - 5 + (1+1+1) 'min'
  - memory retrieval: 8
- Note: Characteristic traces in reaction times!
- Multiple strategies may co-exist (Siegler & Jenkins, 1989) and be used prior to conscious awareness (Siegler & Stern, 1997)

The Problem-Size Effect

- Basic finding (e.g., Campbell, 1995; Zbrodoff, 1995; Geary, 1996): Problems with smaller sums (products) are solved faster, e.g., '4+5' slower than '4+3'
- Explanations are controversial: strategies vs. representation
- Averaging over different strategies may obscure or inflate effects, e.g., decomposition of 6+7 into (6+4)+3 or retrieval vs. counting depending on operands (Siegler, 1987; LeFevre et al., 1996)

Hindu-Arabic Decimal System

- Economy: 9 digits (denoting magnitude) & simple rules
  - position represents a digit's order of magnitude
  - special symbol (0) denoting the vacancy of a position

  => numbers as abbreviated polynomials:

$$9604_{10} = 4 \cdot 10^0 + 0 \cdot 10^1 + 6 \cdot 10^2 + 9 \cdot 10^3$$

- Why 10? Anatomical accident? (See Ifrah, 1994/2000, for alternatives & advantages of base-11 and base-12 ...)

Exp. 1: Serial Addition

- Terminology: u + a = s

  augend addend sum

- Task analysis yields 4 addition types:

Table 2.1: Classification of addition types for adding a single-digit addend a to a single- or double-digit augend Au (u being the augend's unit).

Addition Type		Task Features		
Name	Notation	u	$u+a$	Example
post-increment	[−]	$=0$	<10	10+5
subcomplement	[<]	>0	<10	15+1
complement	[○]	>0	$=10$	15+5
supercomplement	[>]	>0	>10	15+6

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A Serial Addition Paradigm

- Task: Sequentially adding a list of single-digit addends

	4	8	3	5	2	6	…
=> intermediate sums:	4	12	15	20	22	28	…
Addition types:	–	>	<	o	–	<	…

Presentation: 1 addend at a time; press key to see next addend

Materials & Procedure

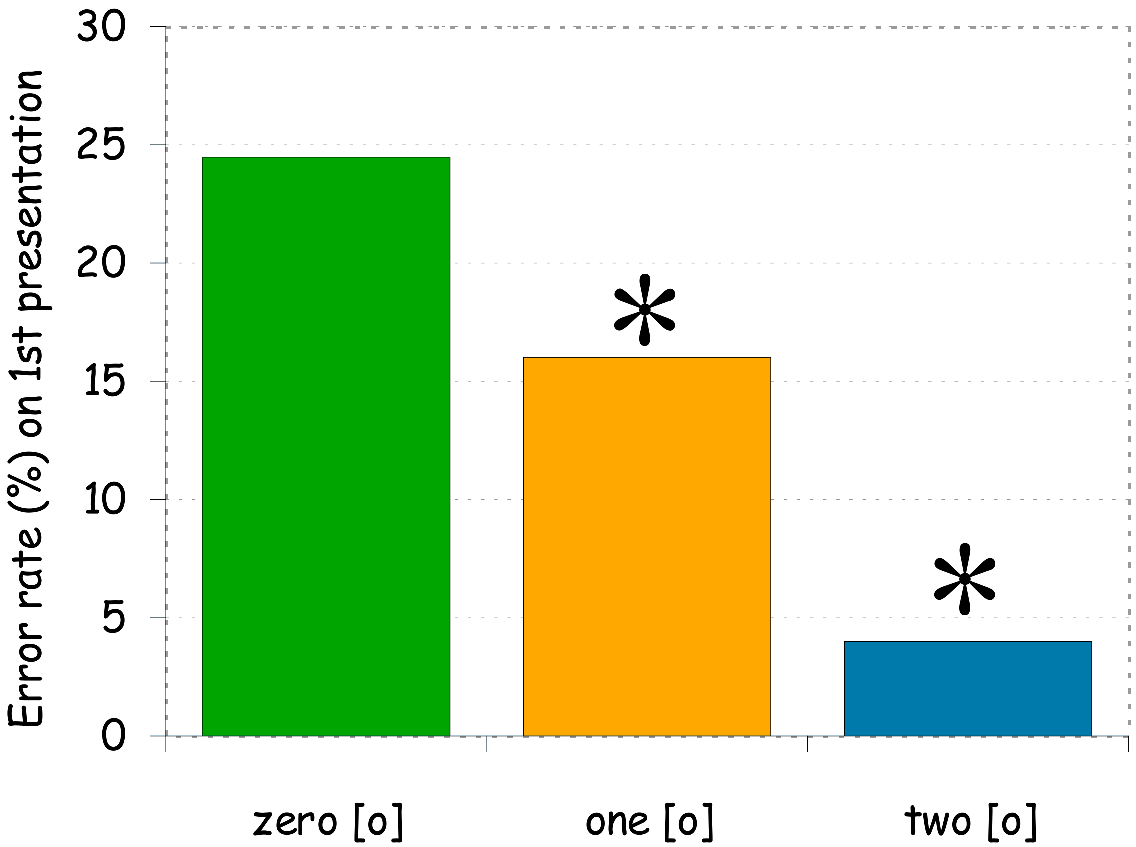
- 20 participants each added a unique set of 30 'random' (but constrained) lists of 4–6 single-digit addends:
  - 10 lists with no complements, e.g. 2 9 5 6 2 5
  - 10 lists with one complement: 2 9 4 5 3 6
  - 10 lists with two complements: 2 9 4 5 3 7
- Participants instructed to add as quickly as possible and enter the result upon a prompt.
- Tacit re-presentation of erroneous trials
  => 30 correct lists/participant

Hypotheses

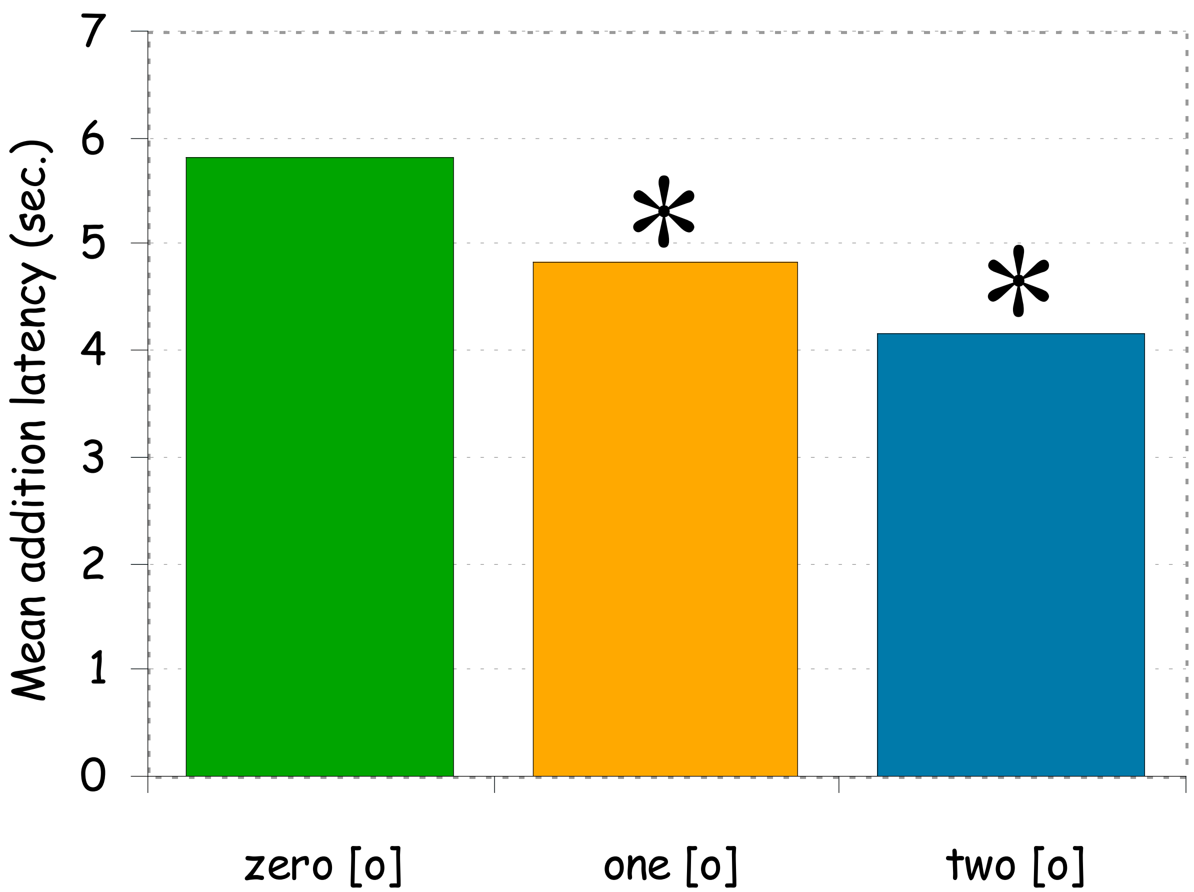
- List level:
  Facilitation by complements, i.e., lists with more complements are added more quickly and/or elicit fewer errors
- Addition level:
  Decade effects, i.e., addition types
  [–] and [o] are added faster than [>] and [<]

Results: List Level

Accuracy:

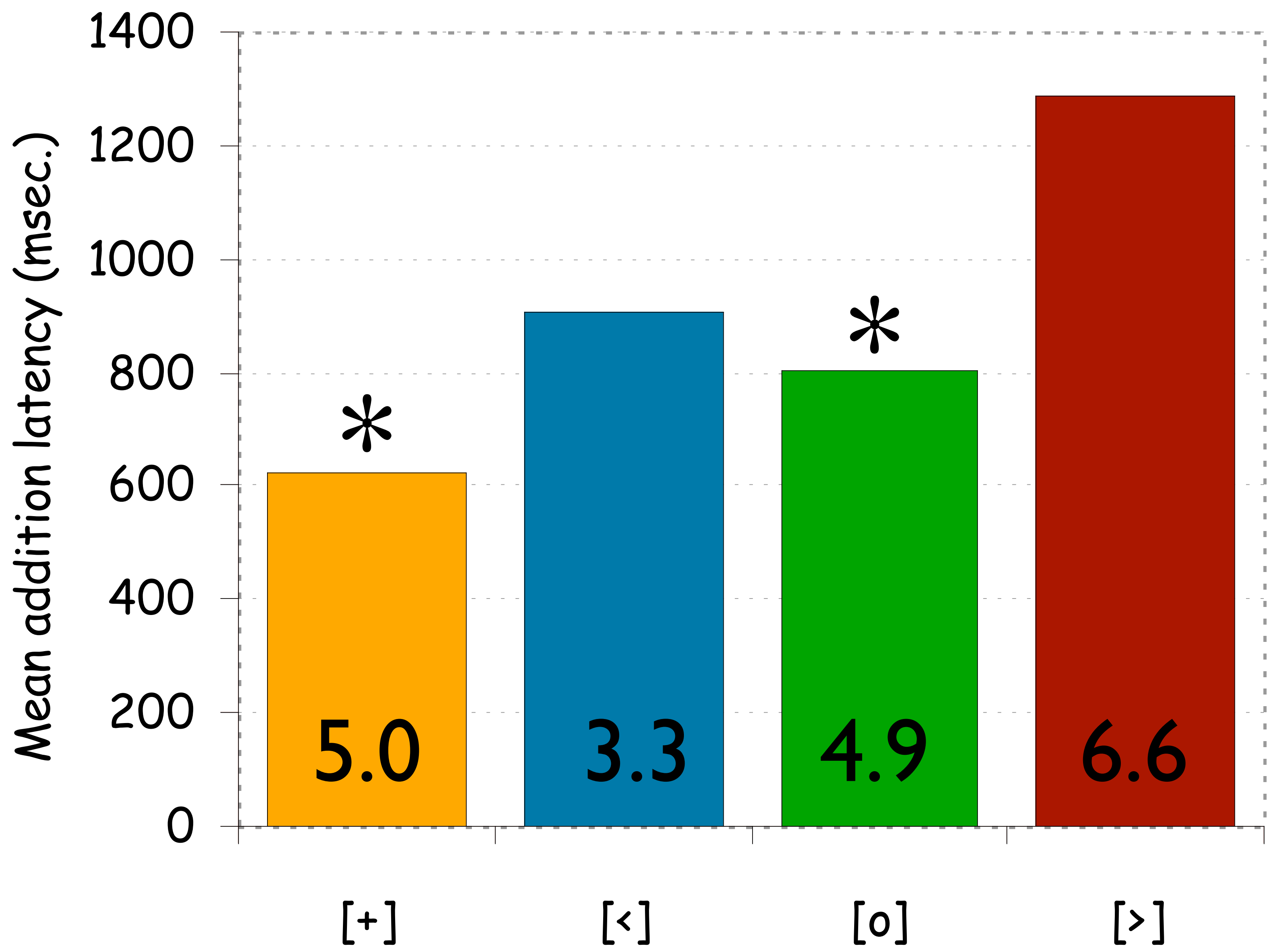


Latency:



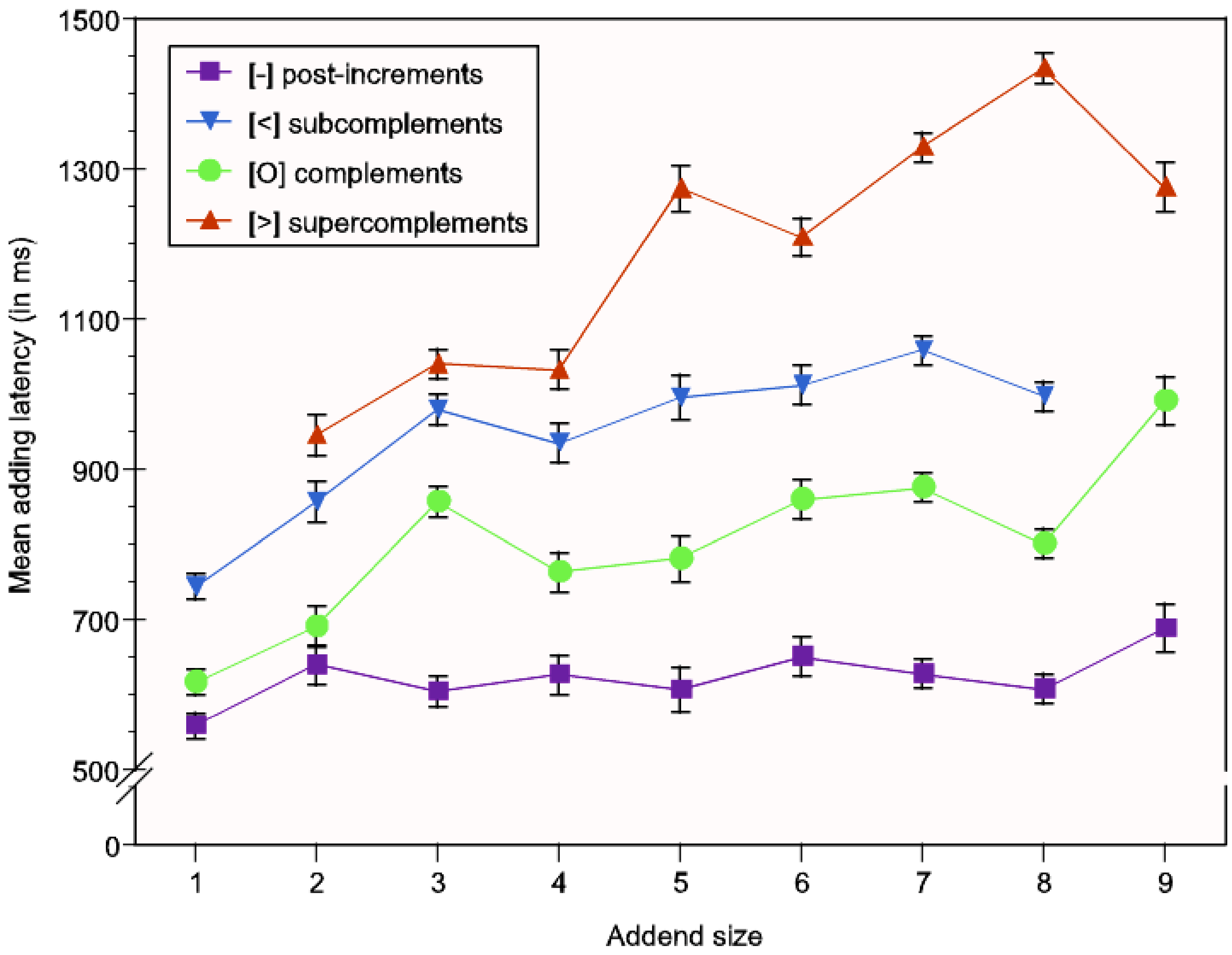
Ambiguity: Facilitation could be due to [o] or [–]…

Results: Addition Level



Caveat: Are types confounded with problem size?

Problem Size Effects?



=> Problem size effects within addition types [<, o, >]

Exp. 1: Discussion of Results

- Notational (decimal) effects:
  - Lists containing complements are added faster & more accurately
  - Double-benefit: [o] and [+] both faster than [<] and [>]
- Note: Intermediate sums were not written in any notation (presumably represented verbally?)
- Questions:
  - How evolved? (Instruction vs. adaptation vs. side effect of frequent decomposition)
  - Actively sought when given a choice?

Exp. 2: Serial Addition of Pairs

- Paradigm: As before, but 2 simultaneous addends => some discretion about the order of operations

$x + a_1 + a_2 = (x + a_1) + a_2$ linear sequence

$x + a_1 + a_2 = (x + a_2) + a_1$ commutativity

$x + a_1 + a_2 = x + (a_1 + a_2)$ associativity

Exp. 2: Addition Types

- New (facilitative?) addition types:

Covert complements:

a) 15 + 2+3 [<o]

b) 18 + 5+7 [>o]

Overt complements:

a) direct: 15 + 5+4

b) indirect: 15 + 4+5

c) pair: 12 + 6+4

d) direct complement-pair: 14 + 6+4

e) indirect complement-pair: 14 + 4+6

Exp. 2: Results & Discussion

- Facilitative effects of (overt & covert) complements:
  - little impact of addend order
  - large effects of opportunity

- Again: Effects of notation on mental processes
- Adders adaptively use minute differences in difficulty to adjust the order of their operations.
- Question: Other 'external' influences?

Exp. 3: Interactive Addition

- "Digital calculations": People routinely manipulate symbols (with hands, pencils, etc.)
- Paradigm: Adding lists of addends or sets of coins

5

12 21

18

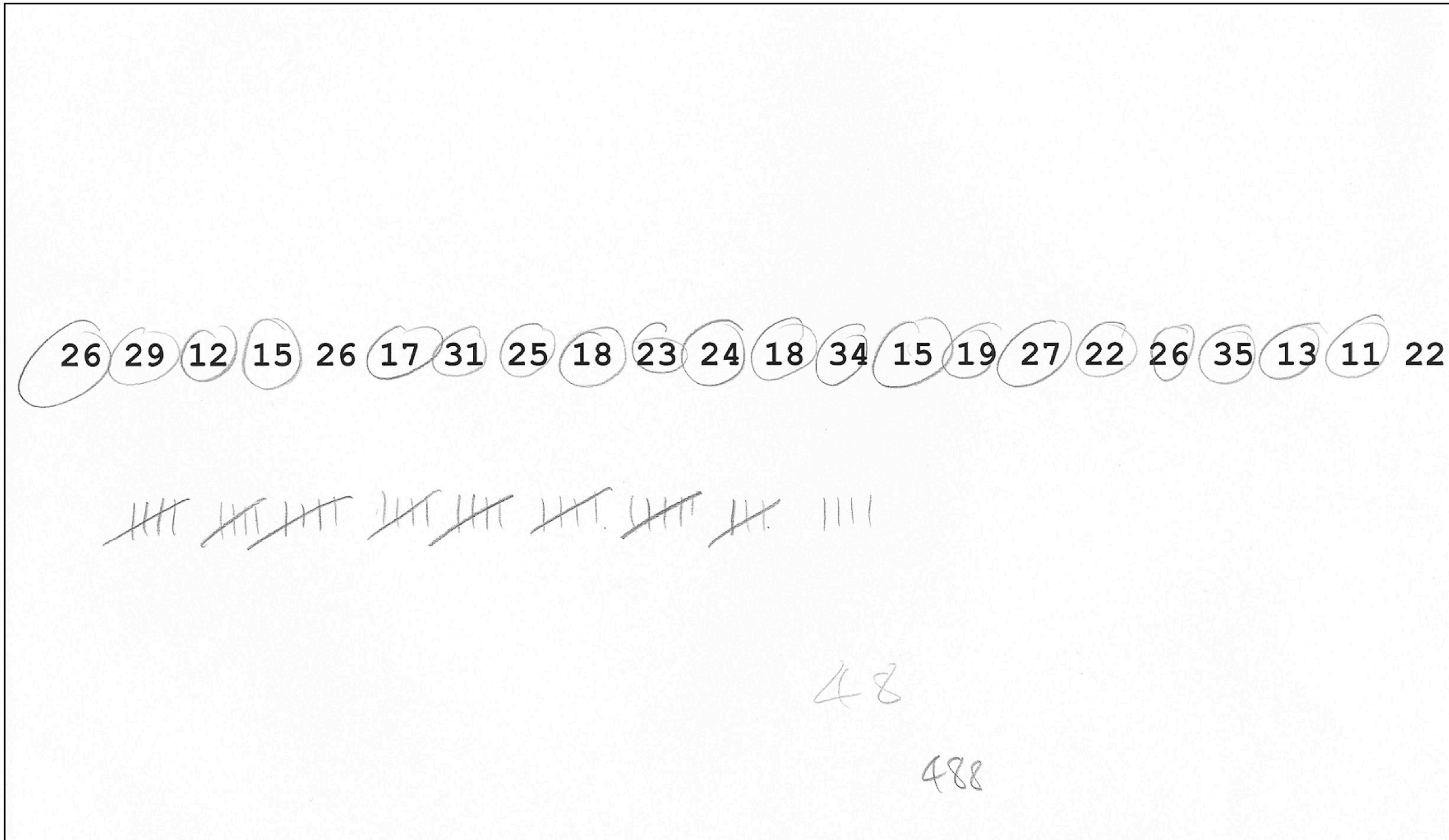
7

9 4

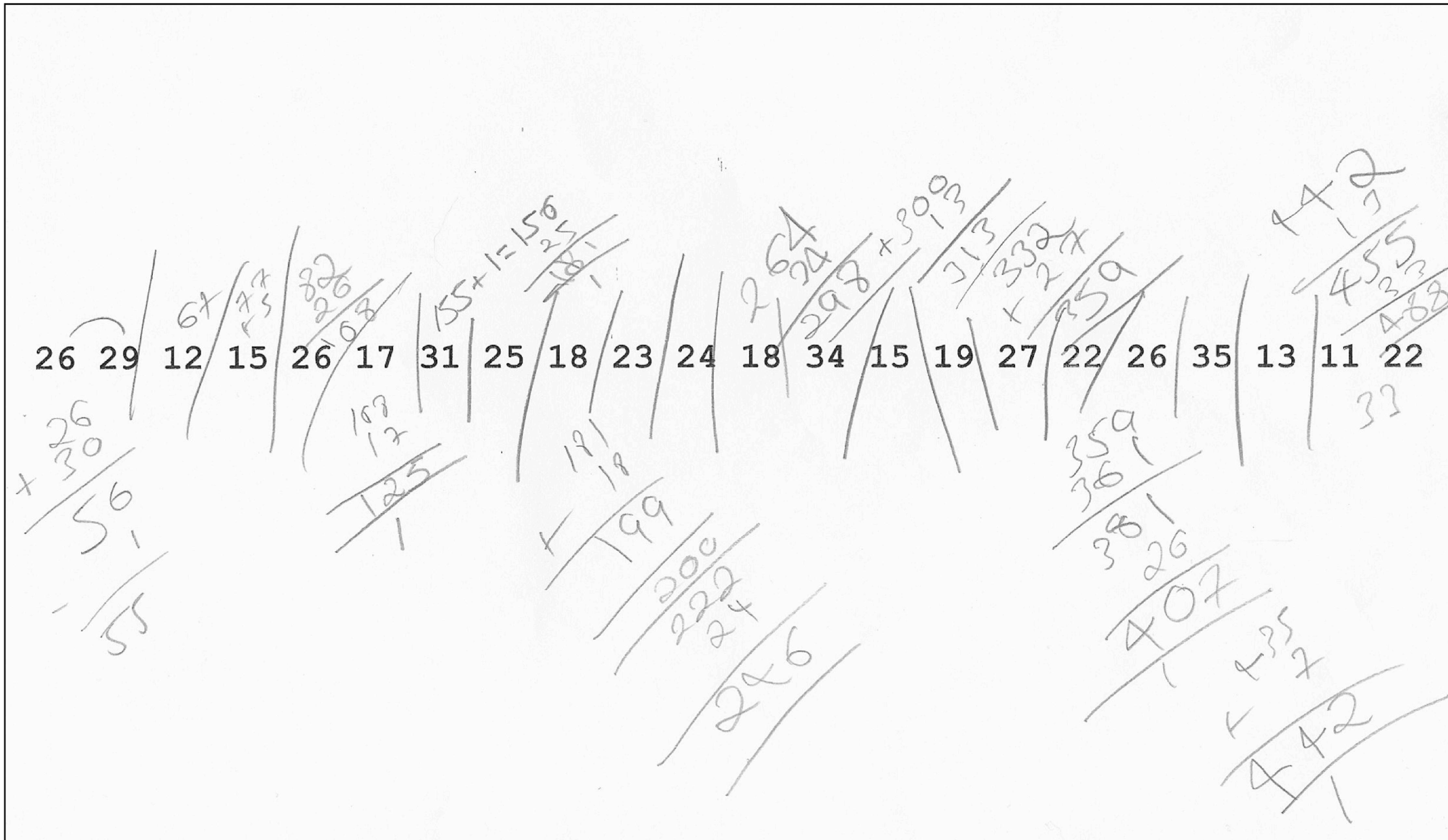
26 17

Use of complementary strategies (Kirsh, 1995)?

Example: Write-to-mark & -tally

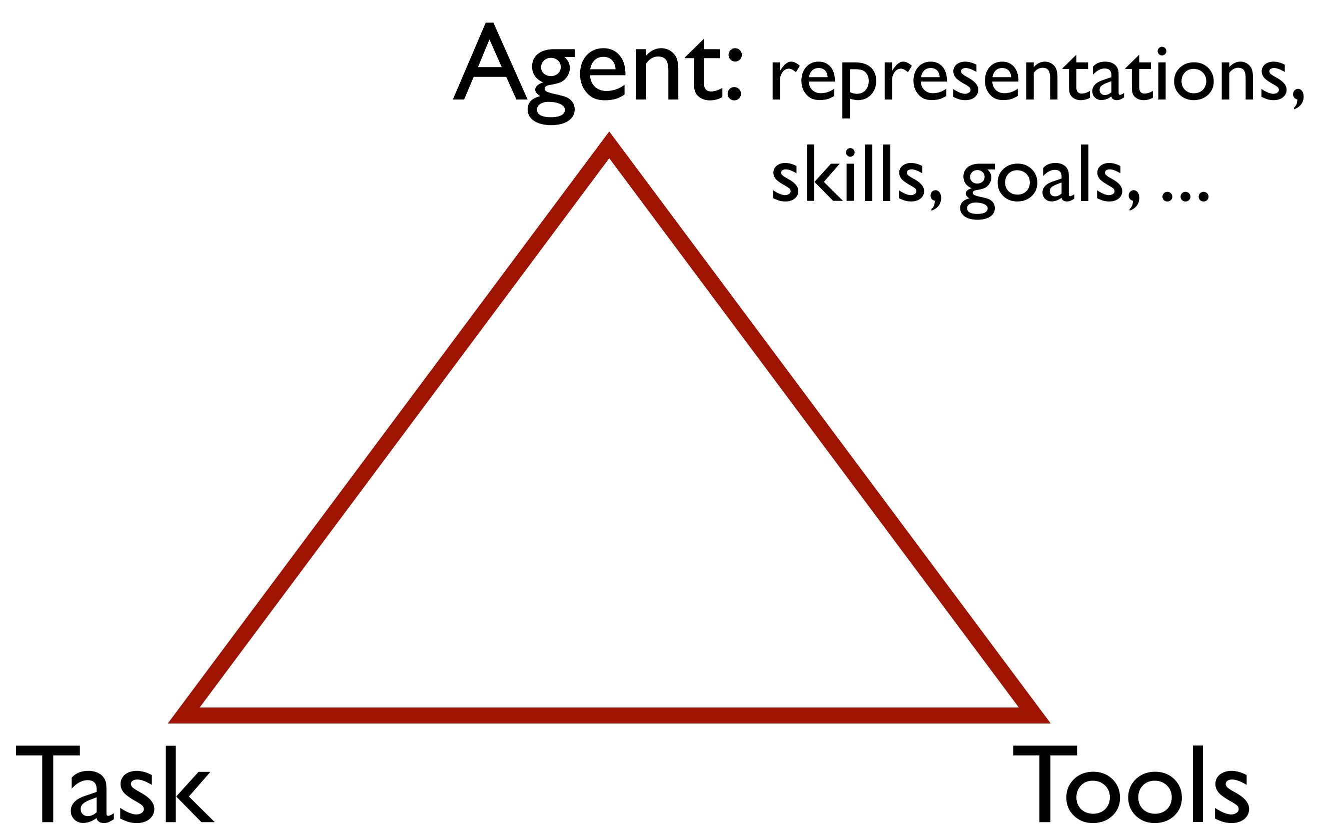


Example: Write-to-mark/add/store/...



Case Study 1: Conclusions

- Notation (external representation) affects mental operations and overt behavior
- Strategies observed depend on (availability, costs and benefits of) interactive resources (tools)
- Adaptive tool use: Type and amount of tool usage is sensitive to agent skills, task properties and usage costs



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Case Study 2: Space Between Symbols

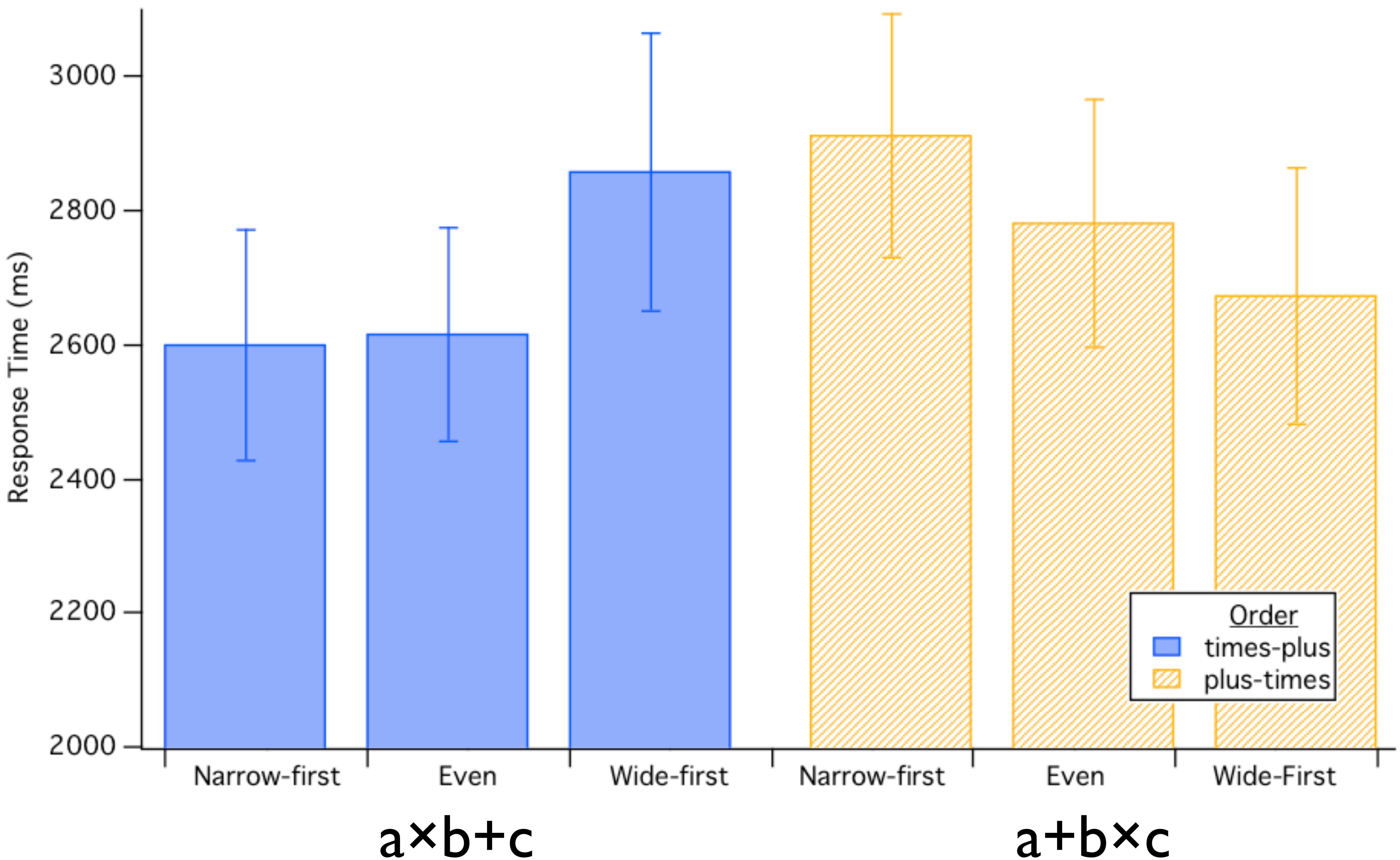
- Landy & Goldstone (2007a, b, c): The Alignment of Ordering and Space in Arithmetic Computation
- Examples: $2 + 4 \times 7$ and $2 \times 4 + 7$
  - formal properties: meaning of symbols, syntax rules
  - accidental properties: font, color, similarities, spacing, ...
- Hypotheses:
  1. operator feature hypothesis: spacing > operator selection
  2. proximity-precedence alignment hypothesis: closer objects are combined first (consistent vs. inconsistent vs. neutral)
  3. expression reading hypothesis: $\times +$ easier than $+ \times$

Case Study 2: Method

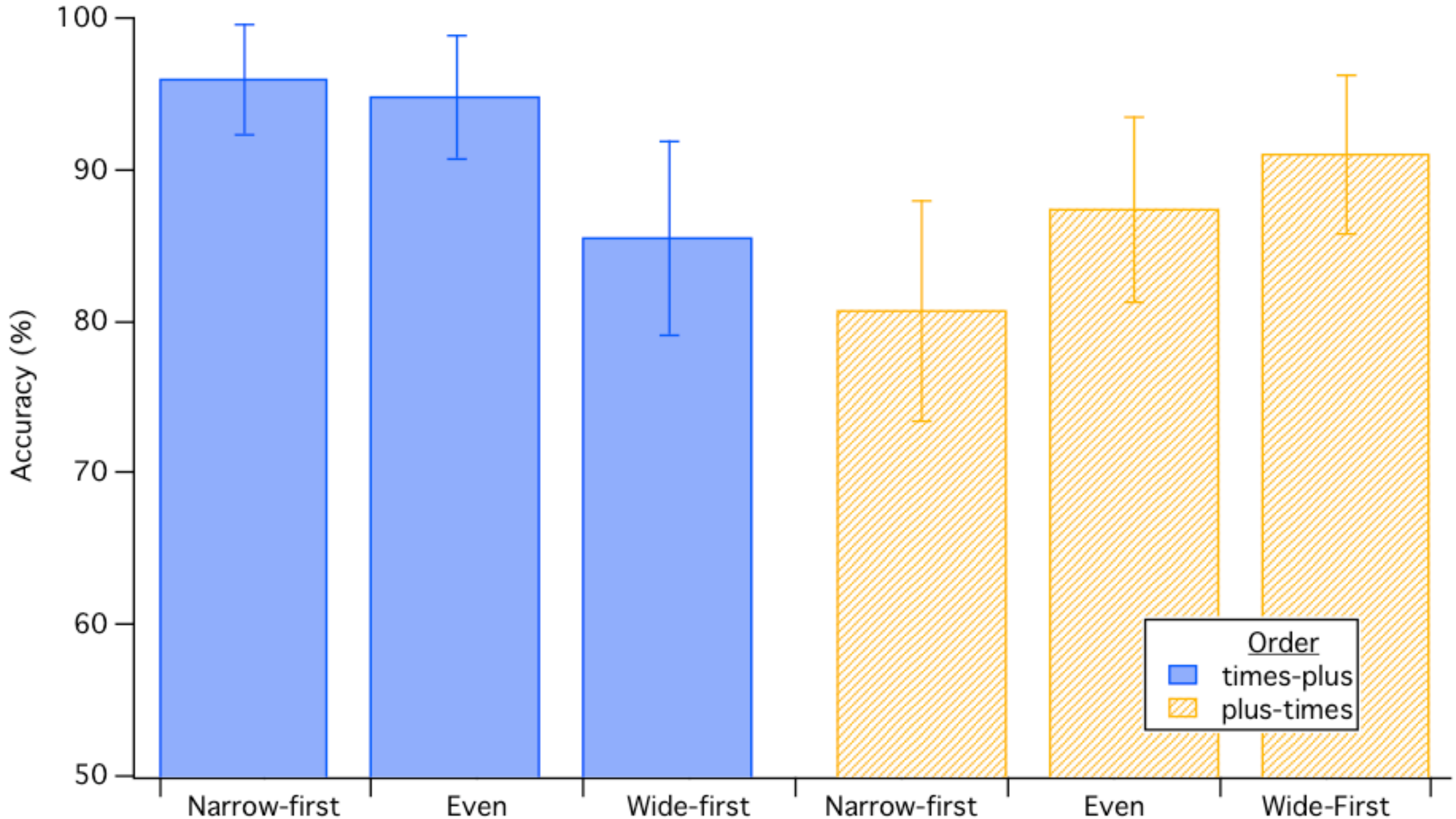
- Materials: a×b+c and a+b×c for a, b, c = 2, 3, 4
- 3 spacing conditions:
  - even: a × b + c   a + b × c
  - narrow-first: a×b + c   a+b × c
  - wide-first: a × b+c   a + b×c

- Task: Calculation as fast as possible; self-paced.

Results: Latency



Results: Accuracy



a×b+c　　　　a+b×c

Results: Specific Errors

- Error Types:
  - — operator confusion
  - — operand errors
  - — precedence errors
- 549 of 971 errors could be classified

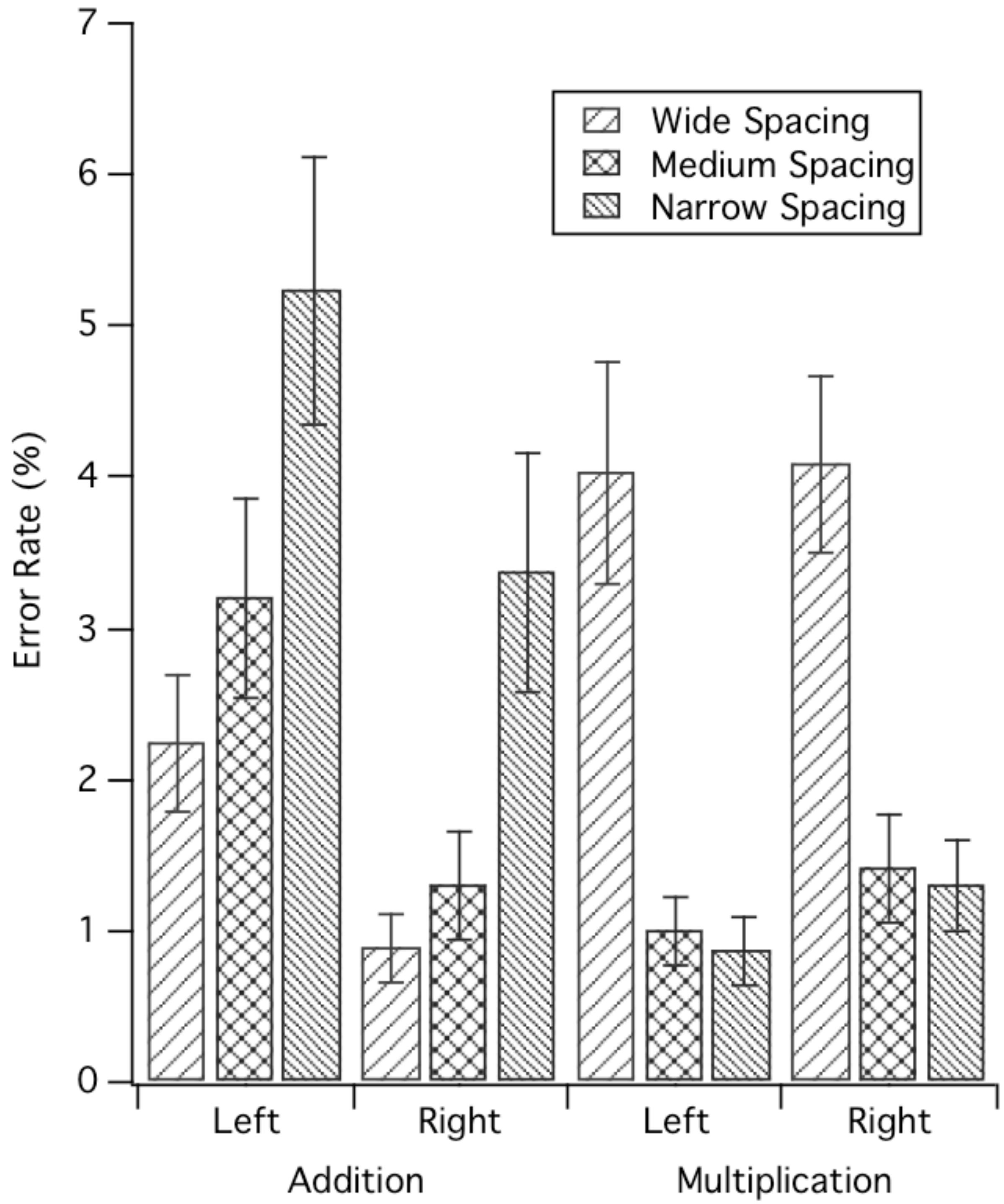


Figure 3: Rate of operator confusion errors (in percentage of all responses), for each operator.

Case Study 2: Conclusions

- Physical properties of notations matter.
- Specifically, spatial layout has various effects, e.g., alignment of spacing with meaning affects task difficulty (even if expression is parsed correctly!)
- Different levels:
  - space affects interpreted meaning of symbols
  - spatial proximity suggests operator preference
  - cultural & perceptual-motor constraints: left-right order
- Genesis: Universal laws vs. familiarity with conventions?

Case Study 3: Strategy Shifts

- Siegler & Stern (1998): Conscious and Unconscious Strategy Discoveries: A Microgenetic Analysis
- Population: 2nd graders (8–9 years)
- Microgenetic method:
  - High density of observations in key phases
  - implicit vs. explicit measures (latency vs. verbal report)
- Inversion task: a + b – b ‘arithmetic insight problem’
  - non-trivial for 1st-4th graders (<50% shortcuts)
  - use of shortcut requires knowledge and recognition

Strategies and Correlates

- What's 18 + 5 – 5?

Table 2
Examples and Definitions of Strategies

		Definition of strategy		
Strategy	Typical overt behavior on 18 + 5 − 5	RT (s)	Explanation	Overt behavior
Computation	"18 + 5—19, 20, 21, 22, 23 [putting up fingers one at a time]—23 − 5; 22, 21, 20, 19, 18 [putting down fingers one at a time]; it's 18"	>4	Computation	Present or absent
Negation	"18 + 5—19, 20, 21, 22, 23 [putting up fingers one at a time]—23 − 5; it's 18"	>4	Negation	Present or absent
Unconscious shortcut	None	≤4	Computation or negation	Absent
Computation–shortcut	"18 + 5—19, 20, 21—oh, it's 18"	>4	Shortcut	Present or absent
Shortcut	None	≤4	Shortcut	Absent

Note. RT = response time.

- Method:
  - 8 sessions: pretest | 6 practice sessions | transfer

mixed vs.
blocked

a+b–b
a–b+b
a+b–a
a–b–a etc.

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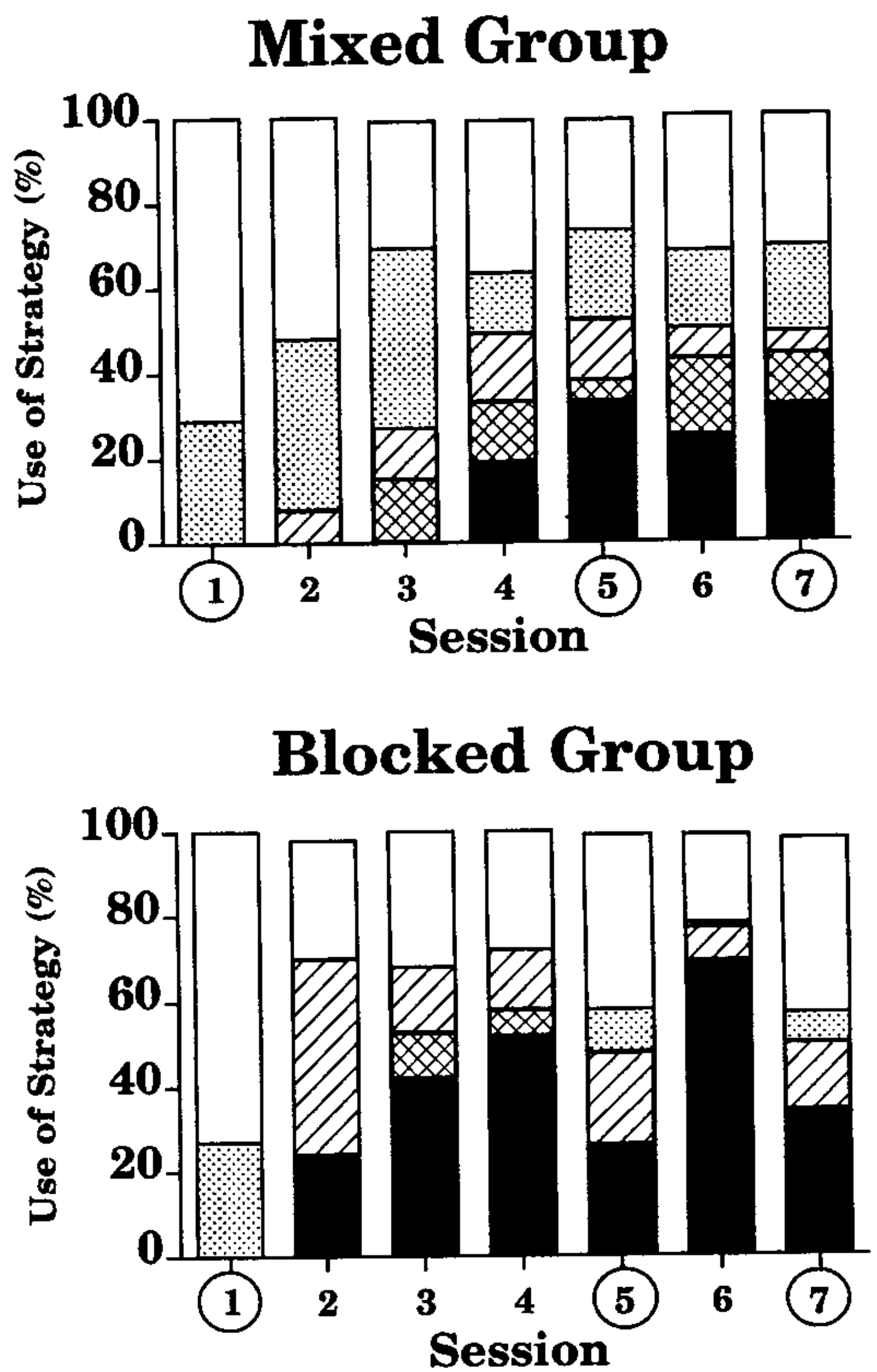
Results: Strategy Uses by Session



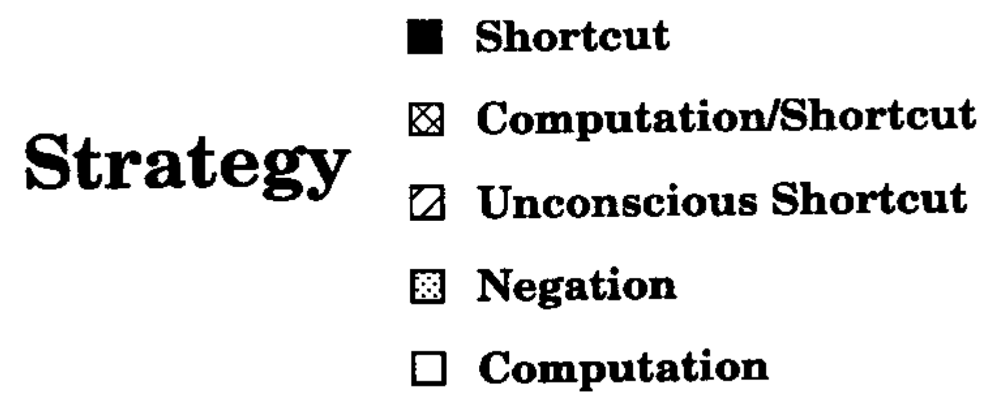


Figure 1. Changes in strategy use over seven sessions. The darker the shading, the more advanced the strategy. Thus, the increasingly dark shading in later sessions indicates increased use of more advanced strategies. Circles around Sessions 1, 5, and 7 indicate that children in the blocked problems and mixed problems conditions received identical problems in those sessions.

Results: Strategy Shift Sequences

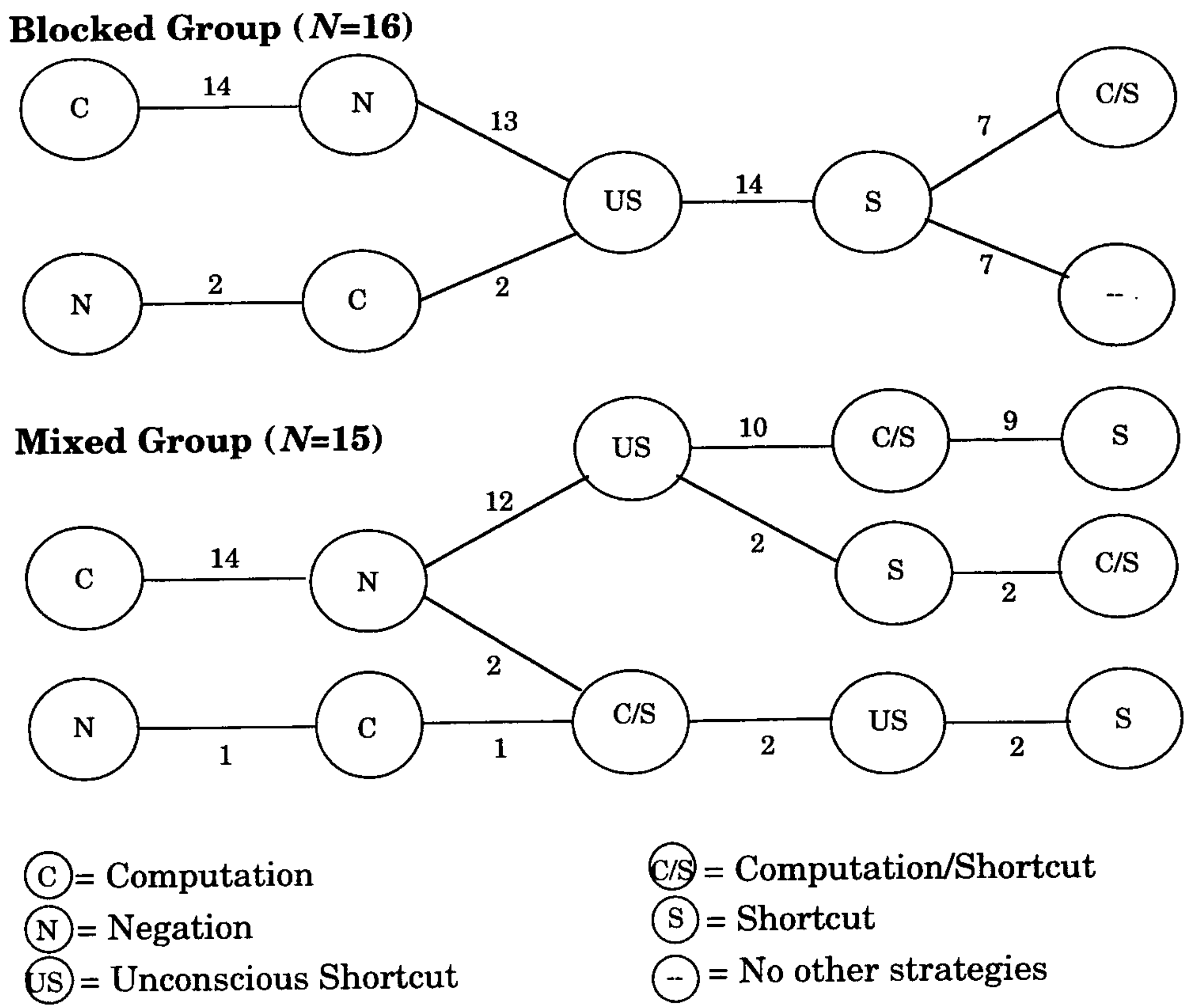


Figure 3. Sequence of first use of strategies. The first strategy used appears on the left; the last strategy used appears on the right. Letters within circles indicate the strategy that was used; numbers between circles indicate the number of children in the group who used the two strategies in that order. Thus, the *14* between the *C* and the *N* at the top left indicates that 14 children in the blocked problems group first used the computation strategy and then used the negation strategy. For purposes of simplicity, only the five main sequences of strategy discovery, used by 27 of the 31 children, are shown. This is why the numbers do not always sum to the total number of children in the group.

Conclusions

- Strategy discovery preceded conscious awareness:
  - 90% showed implicit insight before explicit report of shortcut strategy
  - 80% reported insight within 5 trials of its 1st use
- Persistence of immature strategies
  - An adaptation to changing environments?
- Questions:
  - Generality of unconscious strategy discovery?
  - Mechanism of conscious awareness?
- A methodological marvel!

Mathematics at Work

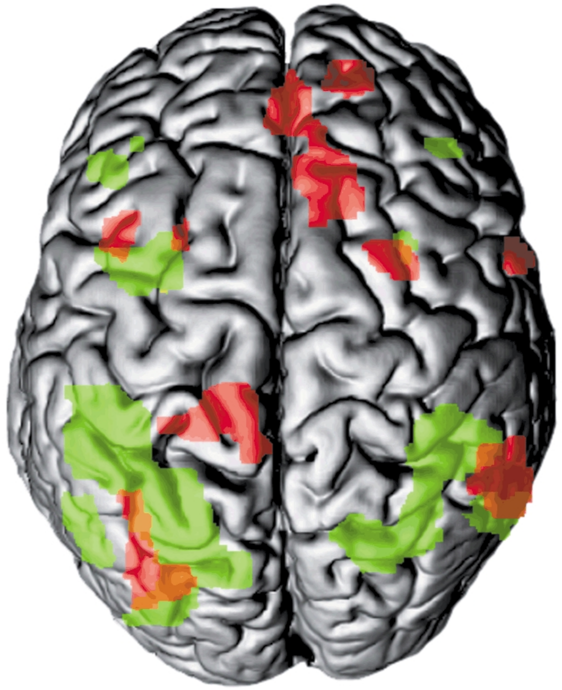
> It is a moot point whether the human hand created the human brain, or the brain created the hand. Certainly the connection is intimate and reciprocal.
>
> A. N. Whitehead

- Mathematics as human practice: trained routines, embedded in contexts, pursuing goals, subject to constraints ...
- Areas & inspirations:
  - Prodigies: Continuum or qualitative leaps?
  - Expertise research: Deliberate practice and routines
- Researching expertise: *How* mathematics is *done*?
  - Tools & devices: Notations, diagrams, proof strategies ...
  - Methods: Verbal protocols, cognitive modeling, brain imaging ...

The Case of Prodigies

- Romantic anecdotes of ‘geniuses’, ‘idiot savants’, etc.
- e.g., Hardy & Ramanujan’s $1729 = 1^3+12^3 = 9^3+10^3$
- Galton’s (1869) ‘hereditary genius’:
  capacity, zeal, and very hard work
- De-mystifications:
  - deliberate training (sometimes pathological)
  - intricate familiarity with facts & methods
  - continuum with normal controls,
    but genuine adaptations
    (e.g., body & brain plasticity,
    LTWM...)

Fig. 1. Top-down view of the brain showing areas that are active in six non-expert calculators as well as Gamm (green), and areas that were specifically active in Gamm (red)[1].



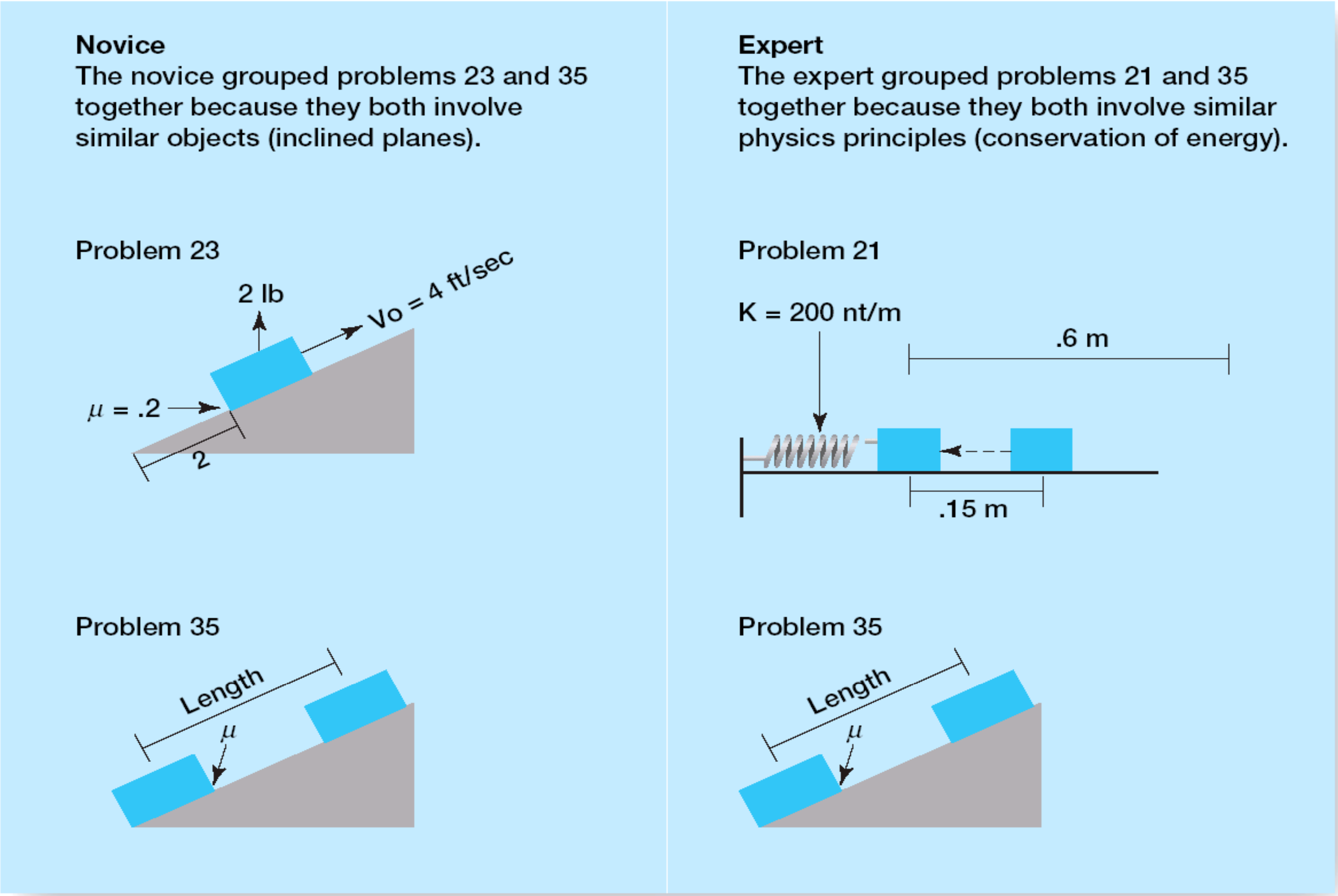
Butterworth (2001), p. 11

Researching Expertise

- ‘Expert’ := ~10+ years of domain experience
- Domains: Math, Chess, Physics, Music, Waiting, Gaming…
- Experts exhibit better (faster & more accurate) problem solving. Questions:
  - Innate talent vs. practice?
  - Same or different mental processes?
- Ericsson et al. (1993, 1994): Deliberate practice, zeal & organizational talent => adaptations

Experts vs. Novices

- Experts have
  - more knowledge (doh!)
    e.g., human calculators know squares, cubes, roots of integers; Chase & Simon's (1973) chess studies
  - differently organized knowledge
    e.g., Chi et al. (1981): Categorizing problems based on surface vs. structural similarities



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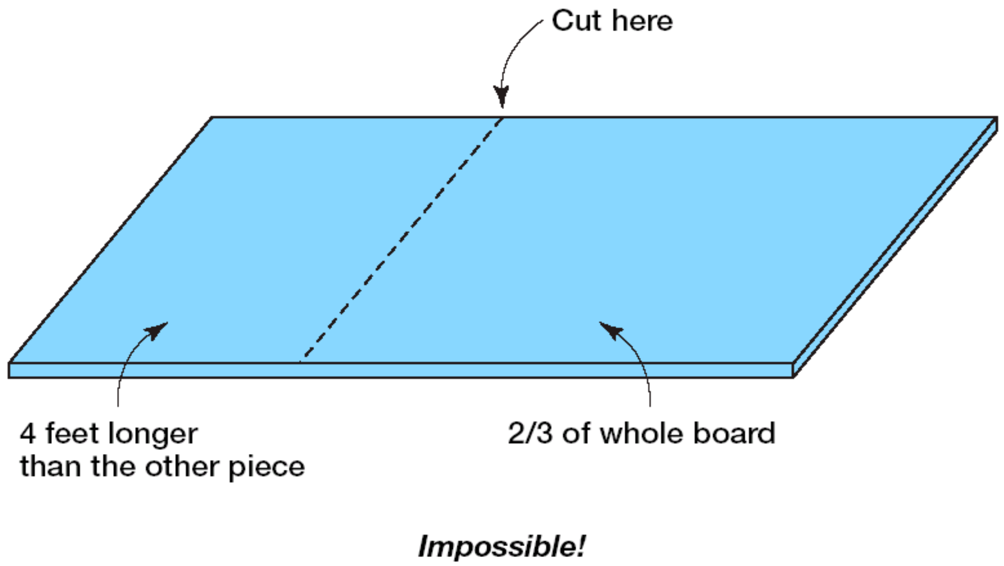
Experts vs. Novices (cont'd)

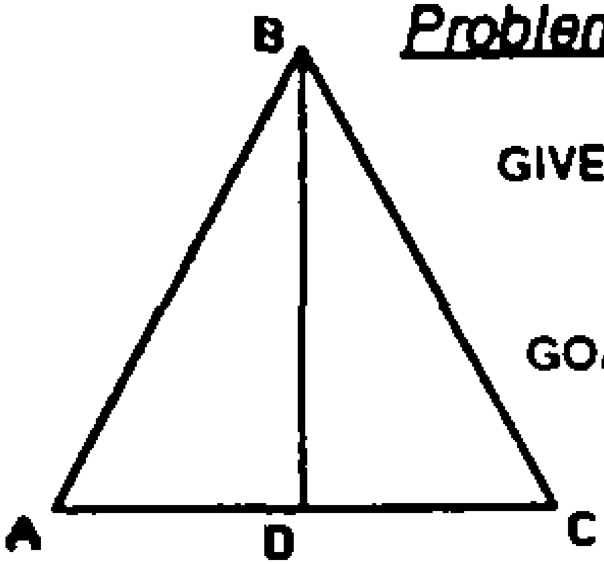
- Experts...
  - use different solution strategies
    (e.g. backwards reasoning; chunking & STM/LTM associations for 79-digit span, Ericcson et al., 1980)
  - Spend more time analyzing, rather than solving a problem
    (Paige & Simon, 1966)
  - But Caveats: Expertise...
    - is domain-specific. Voss et al. (1983): Poor transfer of expertise
    - can result in functional fixedness
      (= blindness to creative alternatives)



Case Study 4: Geometry Proofs

- Koedinger & Anderson (1990): Abstract planning and perceptual chunks: Elements of expertise in geometry
- Diagram Configuration (DC) model, based on earlier geometry tutors
- Assumptions: Geometry experts
  - focus on key steps, skip intermediate ones,
  - parse diagrams into perceptual chunks, and
  - reason on the basis of schematic diagram configurations

Example Task & Proof



Problem 3

GIVENS: **rt ∠ADB**

$\overline{BD}$ bisects ∠ABC

GOAL: **D midpoint of $\overline{AC}$**

B1: We're given a right angle—this is a right angle,
B2: perpendicular on both sides [makes perpendicular markings on diagram];
B3: BD bisects angle ABC [marks angles ABD and CBD]
B4: and we're done.

B5: We know that this is a reflexive [marks line BD],
B6: we know that we have congruent triangles; we can determine anything from there in terms of corresponding parts
B7: and that's what this [looking at the goal statement for the first time] is going to mean. . . that these are congruent [marks segments AD and DC as equal on the diagram].

Planning Phase

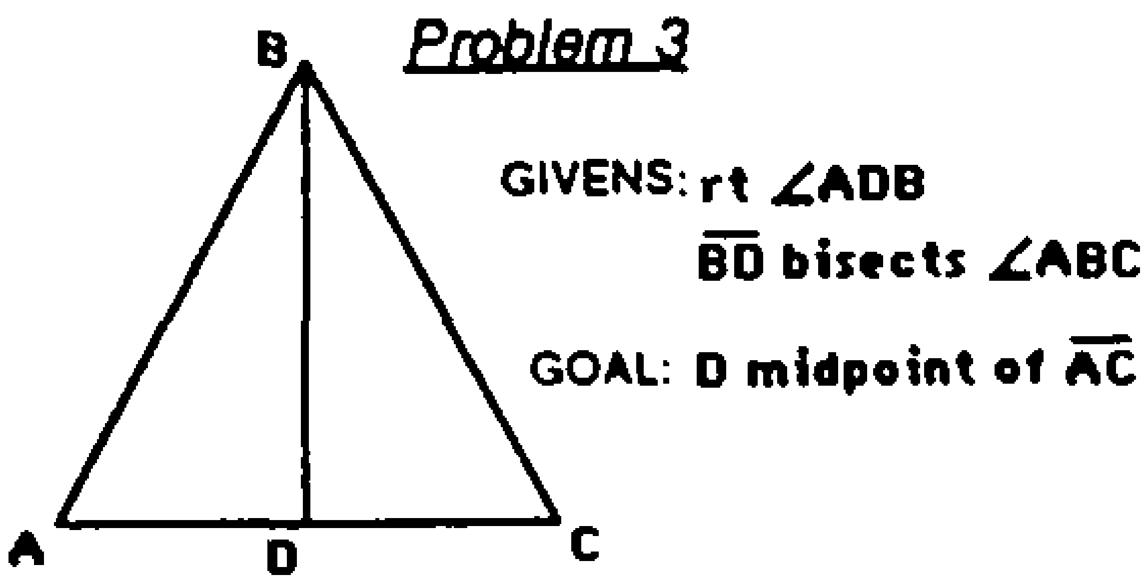
Reading given: **rt ∠ADB**
Inference step 1: **AC ⊥ BD**

Reading given: **BD bisects ∠ABC**
Inference step 2: **△ABD ≅ △CBD**

Execution Phase

In this phase, the subject refines and explains his solution to the experimenter.

Expert's Step Skipping & Abstraction



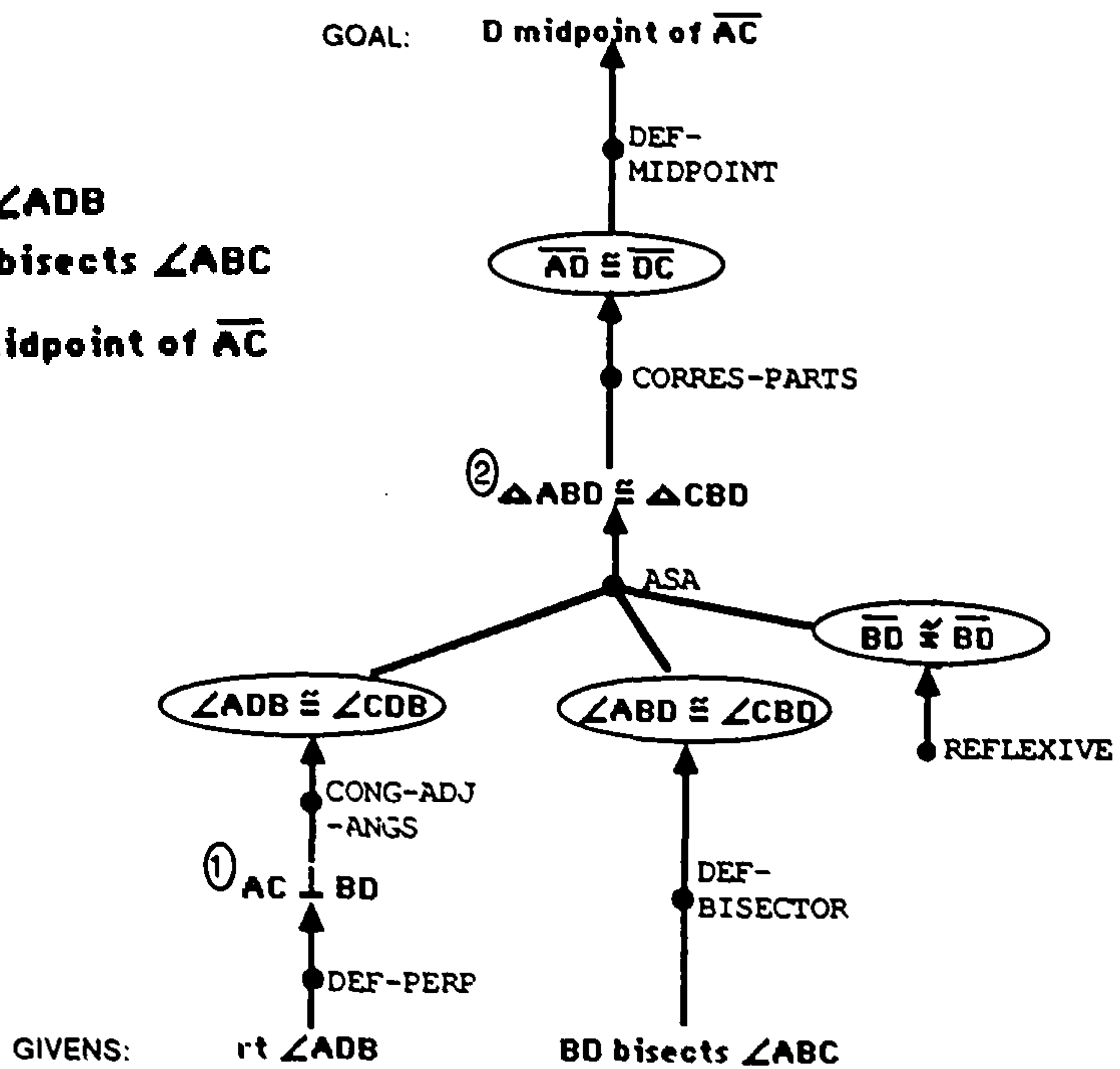


Diagram Configuration Schemas

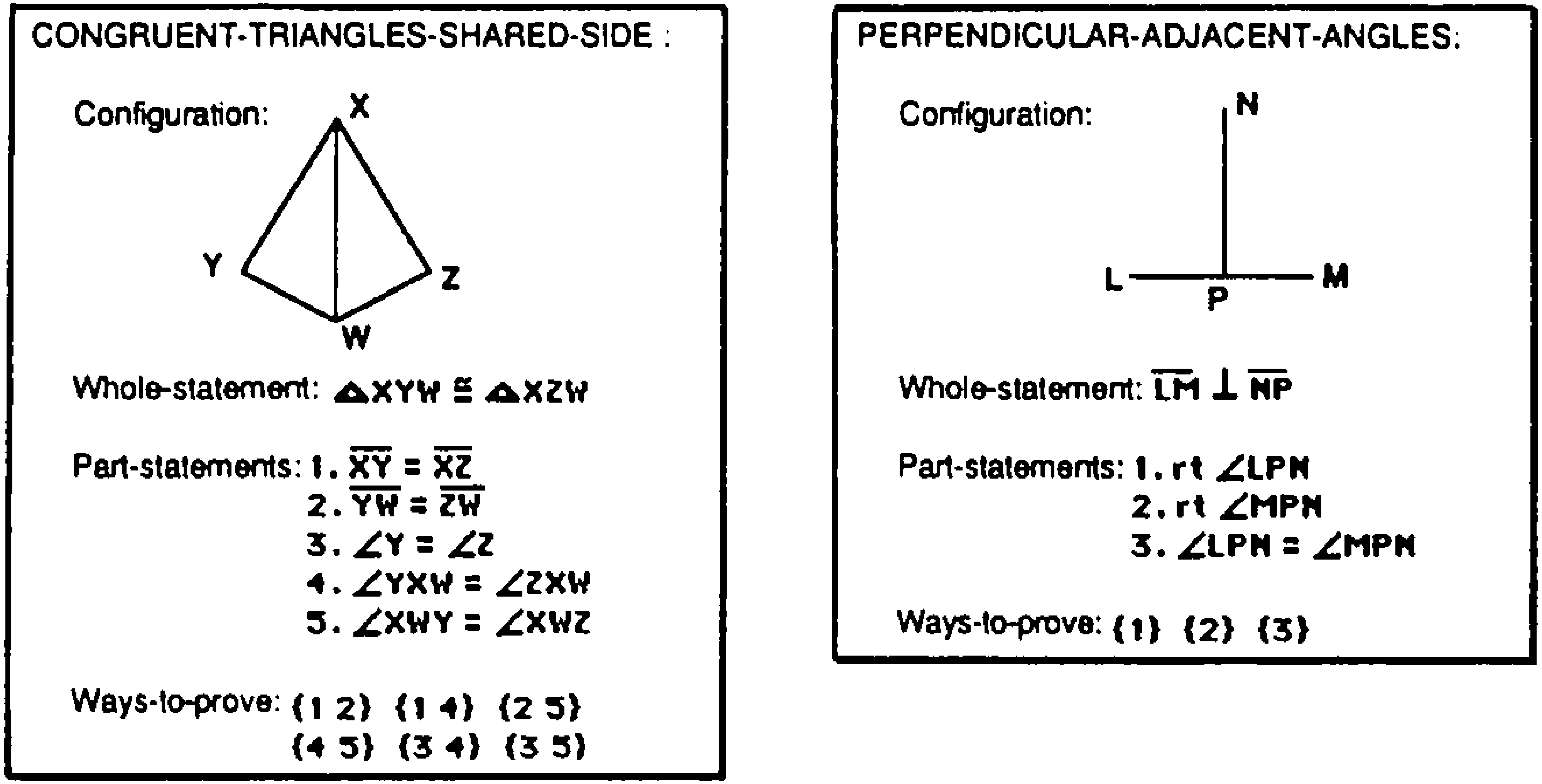
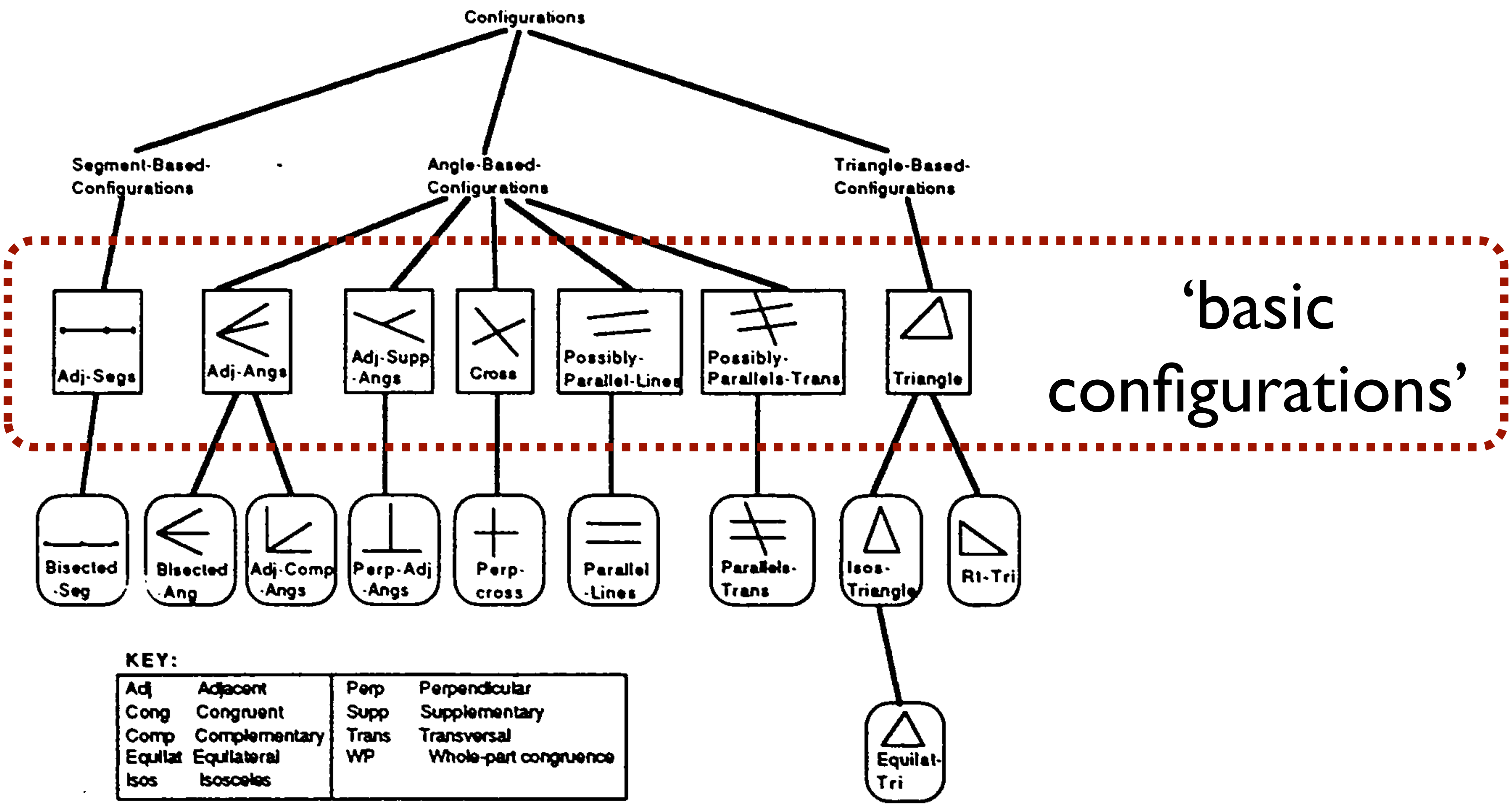


Figure 3. Two examples of diagram configuration schemas. The numbers in the ways-to-prove indicate part-statements. Thus, in the CONGRUENT-TRIANGLES-SHARED-SIDE schema {1 2} means that if the part-statements **XY=XZ** and **YW=ZW** are proven, all the statements of the schema can be proven.

DC's Processing Components

- Planning & search through space of diagram configurations, rather than axioms of geometry
- Processing components:
  - Diagram parsing → Schema instantiation
  - Statements encoding: given- & goal-statements
  - Schema search: forward or backward inferences

Diagram Parsing & Schema Instantiation



- Note: Perceptual processes are not modeled!

Proof by Recognition & Search...

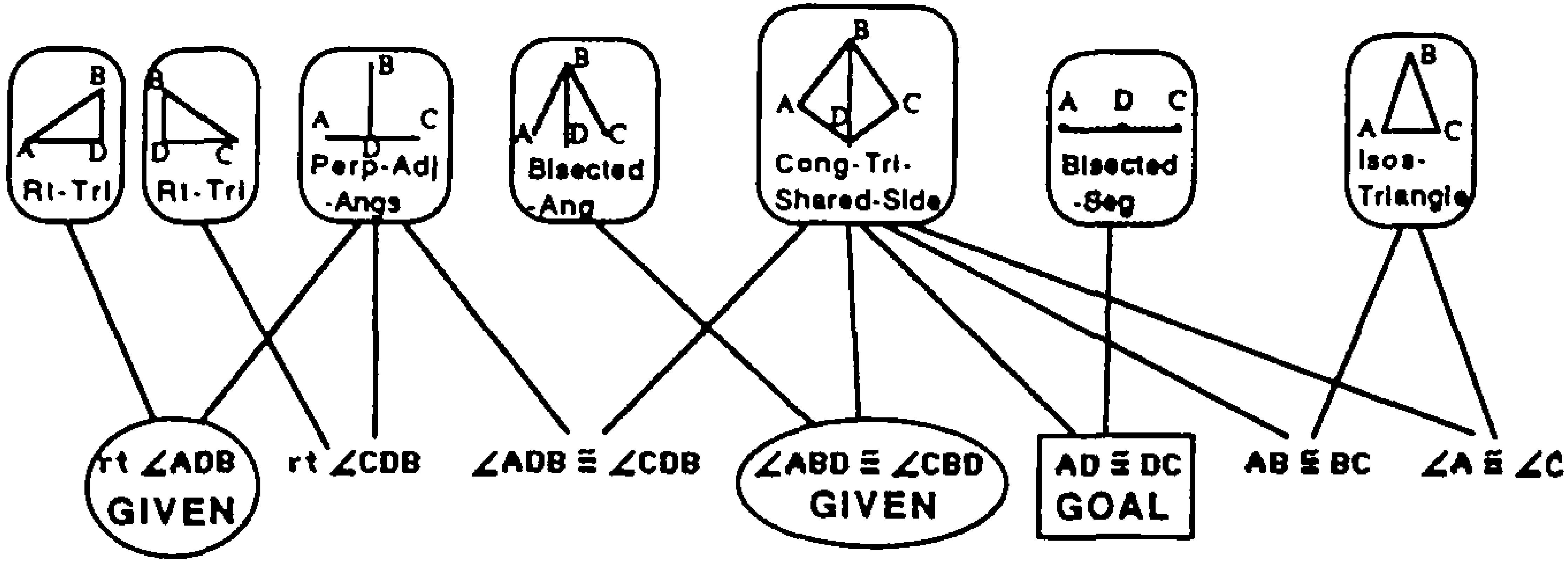


Figure 5. DC's solution space for Problem 3. The schemas DC recognizes during diagram parsing are shown in the boxes. The lines indicate the part-statements of these schemas. A solution is achieved by finding a path from the givens to the goal satisfying the constraints of the ways-to-prove slot of the schemas used.

- Note: 95% is done in phase 1 (schema instantiation)...

Model Evaluation

- Combinatorial analysis:
  - higher ‘effectiveness’ (smaller search space) than alternative models
- Empirical evaluation:
  - N=8 (Geometry teacher, researchers, grad students)
  - Method: Verbal protocol analysis (Ericsson & Simon, 1984) & proof tree diagrams
  - Verbalization assumption: One verbalization per schema application

Evaluation Results

Model-Data Fit for All Subjects Solving Problem 7

	Predicted Mention		Predicted Skip	
Subject	Actually Mention	Actually Skip	Actually Mention	Actually Skip
R	3	0	3	2
B	2	0	1	3
K	3	0	1	6
J	2	0	1	3
F	3	2	3	9
Total	13	2	9	23

Discussion & Conclusion

- Very efficient & good fits to expert behavior.
- The power of representation...

- But: Side-stepping phenomenon (by 'explaining it away')?
  - No explanation of knowledge acquisition: (DCs, declarative → procedural)
  - Perceptual processes not modeled
  - Note parallels to models of human reasoning

Mathematical Skills

- Mathematical learning: training & skill acquisition
- Mathematics as “problem solving”
  (:= what we do when we don’t know what to do)
- Components
  - representation (states)
  - memory for facts vs. procedures
    (knowledge & operators)
  - control strategies: maintaining goal hierarchies, minimizing memory load etc.

Case Study 5: Algebra & ACT-R

- Anderson (2005): Human Symbol Manipulation Within an Integrated Cognitive Architecture
(see also Anderson, 2007)
- Task domain: Simple algebra expressions, e.g. $7x+3=38$
- ACT-R (“adaptive control of thought-rational” analysis)
  - comprehensive psychological theory (architecture)
  - programming language for cognitive modeling
  - framework to organize thought

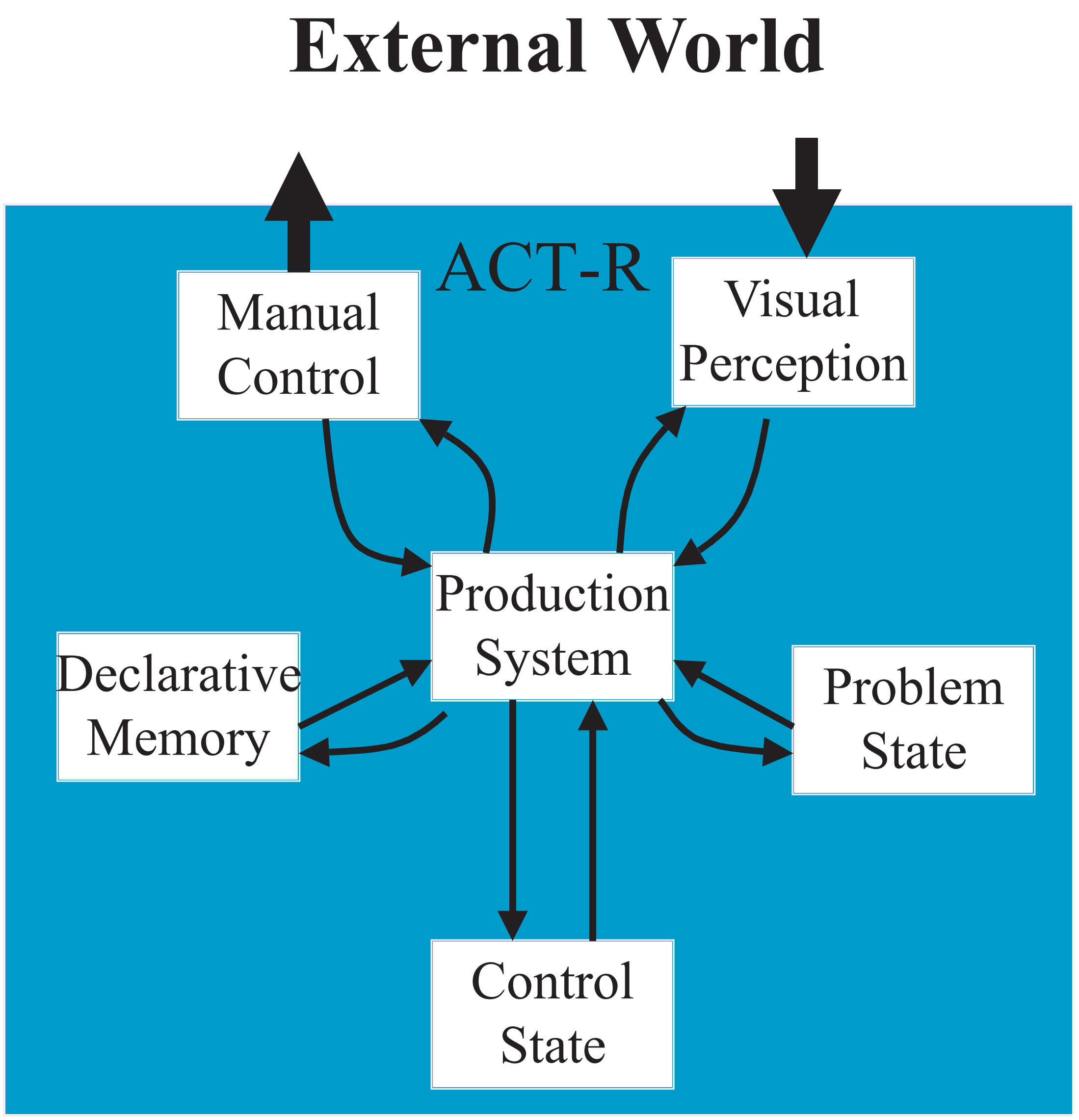
Modules & Buffers

- Modules:
  - visual
  - manual
  - procedural
  - declarative
  - imaginal
  - control (goal)

peripheral

central

- massive parallelism within modules
- communication via buffers
  - contain 1 ‘chunk’



Knowledge Representation

- Declarative knowledge: facts
  - represented as structured chunks
- Procedural knowledge: productions

  IF the goal is to solve the equation
  and the equation is of the form Expression – Number1 = Number2
  and Number1 + Number2 is Number3 has been retrieved
  THEN transform the equation to Expression = Number3

  - test buffer contents, 'pattern matching & manipulation'
  - serial bottleneck (50msec)
- Symbolic vs. sub-symbolic components: structures vs. activations/utilities

Method

- Participants: 10 students (11–14 yrs.)
- Sample tasks:

0-step: e.g., $1x + 0 = 4$

1-step: e.g., $3x + 0 = 12$, $1x + 8 = 12$

2-step: e.g., $7x + 1 = 29$

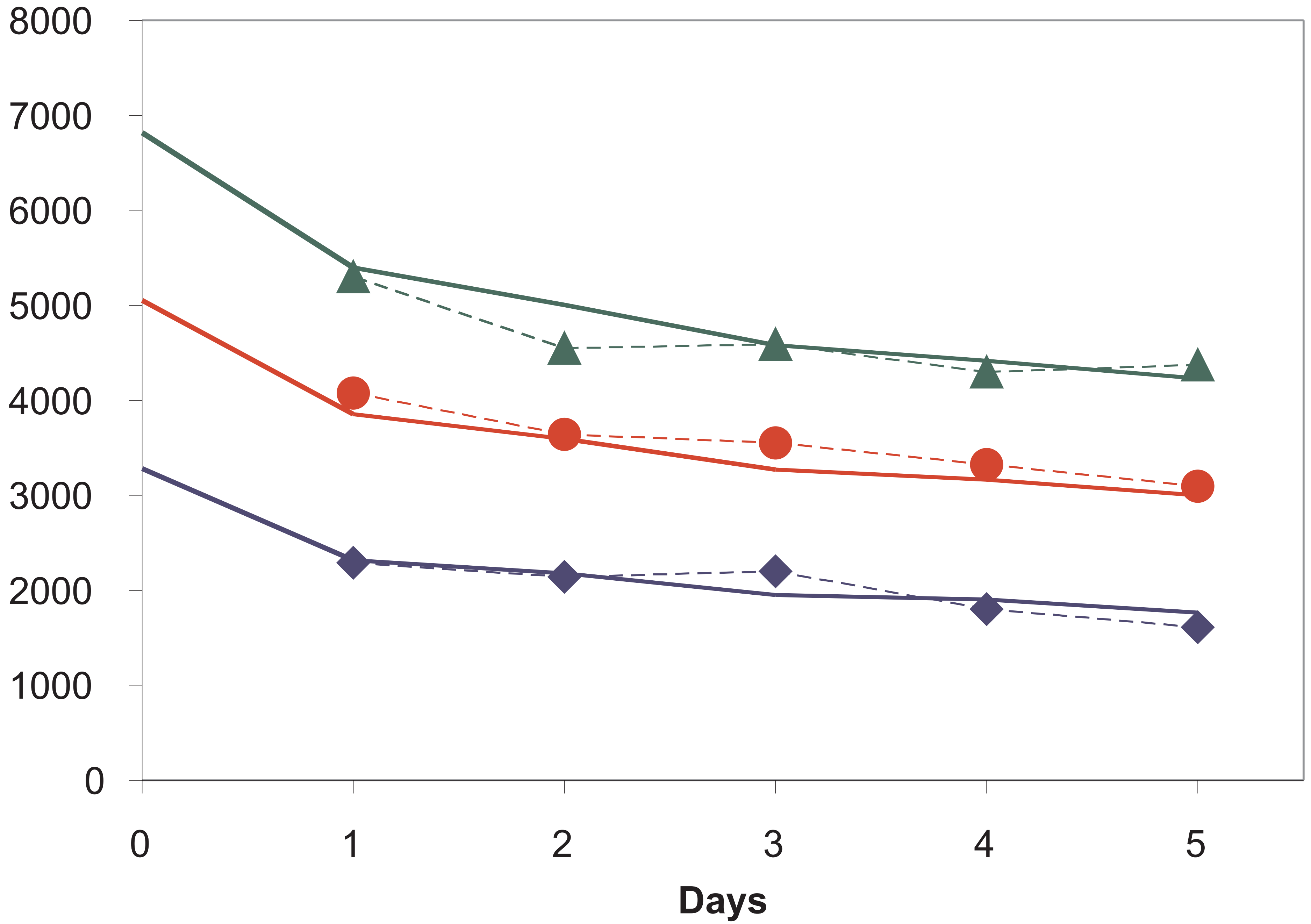
- Instruction & 5 days, 10 x 16 trials per day
- Model predictions vs. empirical data vs. brain imaging

Model Instructions

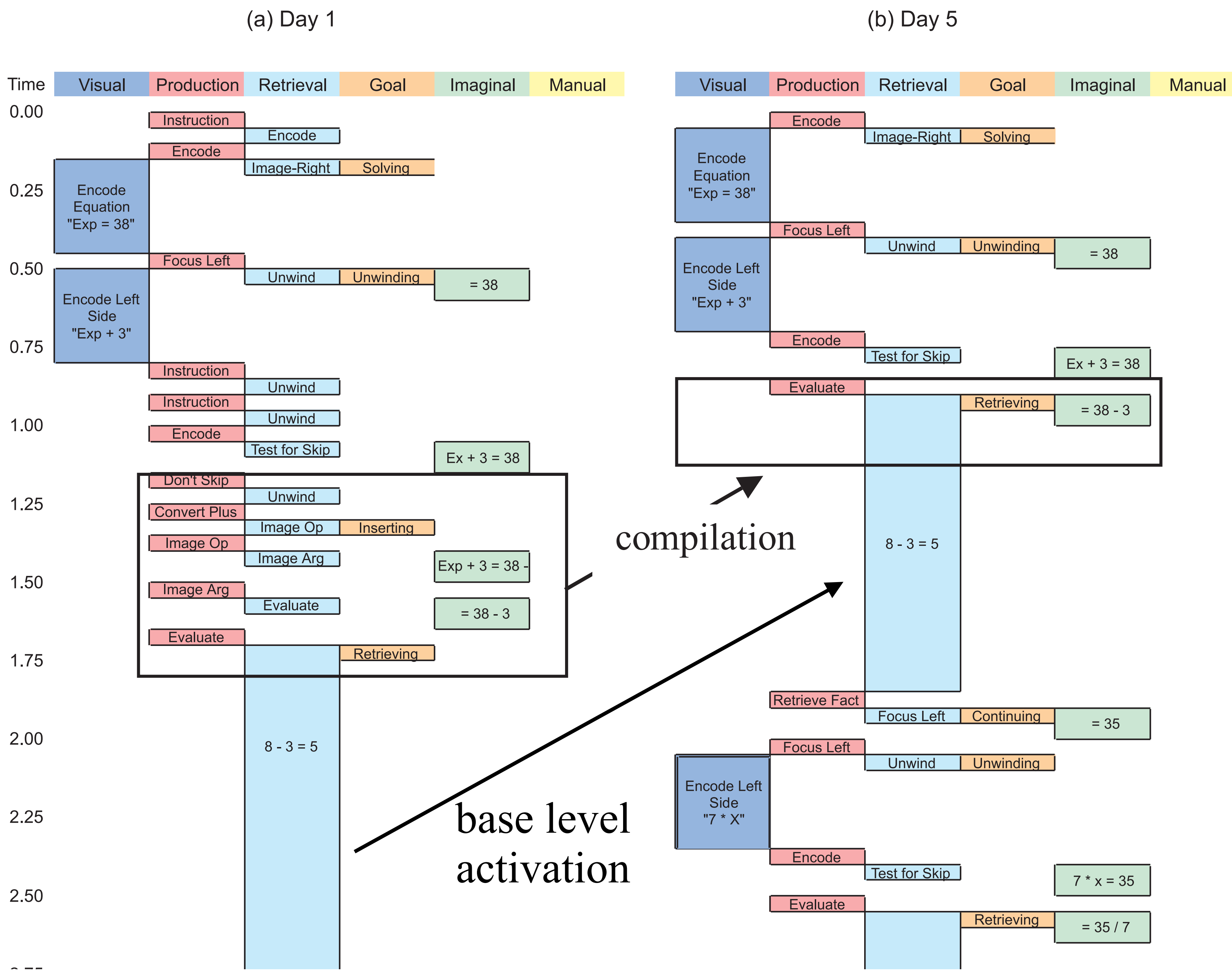
Table 1
English rendition of instructions given to ACT–R model for equation solving

1. To solve an equation, encode it and
   a. If the right side is a number, then imagine that number as the result, and focus on the left side and unwind it.
   b. If the left side is a number, then imagine that number as the result, and focus on the right side and unwind it.
2. To unwind an expression
   a. If the expression is the variable, then the result is the answer.
   b. If a number is on the right unwind-right.
   c. If a number is on the left unwind-left.
3. To unwind-right, encode the expression (of the form "subexpression operator number") and
   a. If the operator is + or – and the number is 0, then focus on the subexpression and unwind it.
   b. Otherwise invert the operator, imagine it as the operator in the result, imagine the number of the expression as the second argument in the result, evaluate the result, and then focus on the subexpression and unwind it.
4. To unwind-left encode the expression (of the form "number operator subexpression") and
   a. If the operator is * and number 1 then focus on the subexpression and unwind it.
   b. Otherwise check that the operator is symmetric, invert the operator, imagine it as the operator in the result, imagine the number as the second argument in the result, evaluate the result, and then focus on the subexpression and unwind it.

Results & Model Fit



Model Trace & Learning



Converging Evidence via Brain Imaging

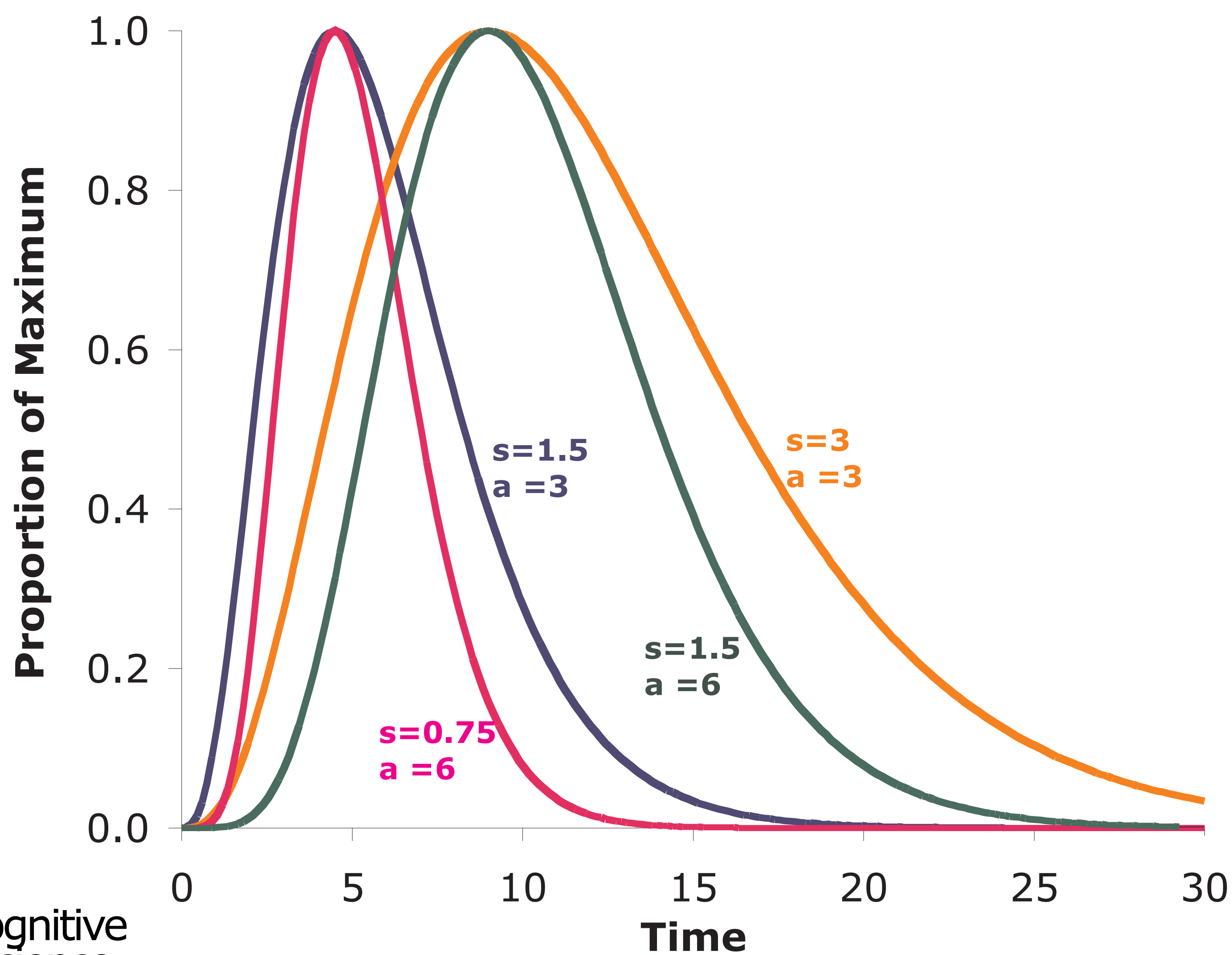
- Postulate cerebral correlate for each buffer (a priori)
- Modules: Brain region:

Modules:	Brain region:
– visual	(visual)
– manual	(motor)
– procedural	caudate (now: basal ganglia)
– declarative	prefrontal
– imaginal	parietal
– control (goal)	anterior cingulate

- Goal: Triangulation of data/model/brain
- Method: fMRI (data glove)

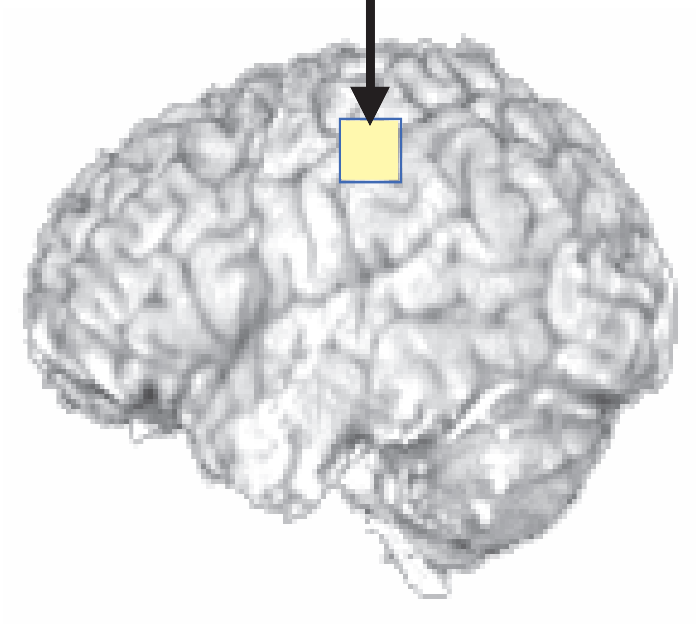
BOLD response

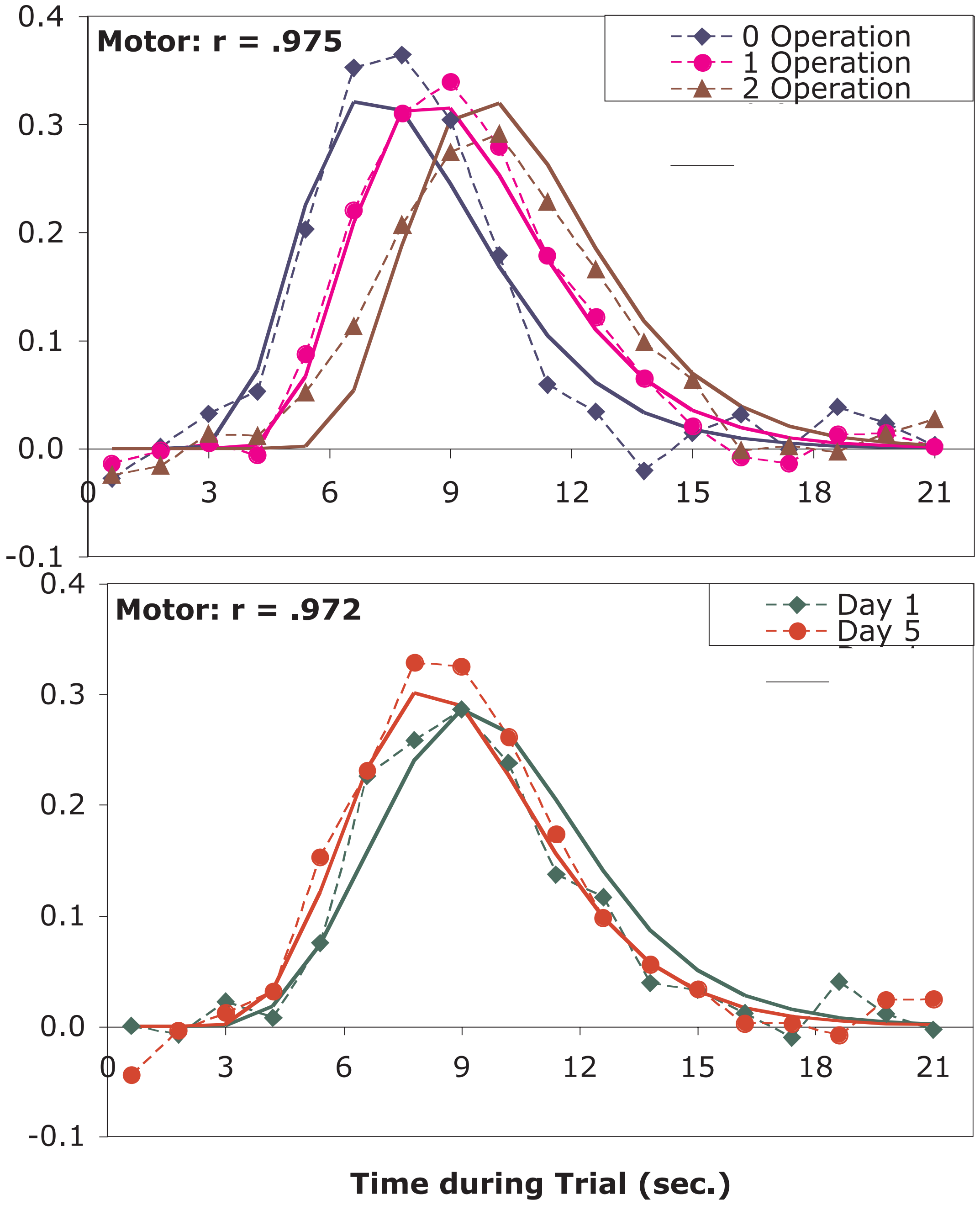
- Blood-oxygen level dependent (BOLD):
  Delayed hemodynamic response indicating summary of metabolic activity
- 3 parameters: magnitude, time scale, shape



Results: Motor/Manual

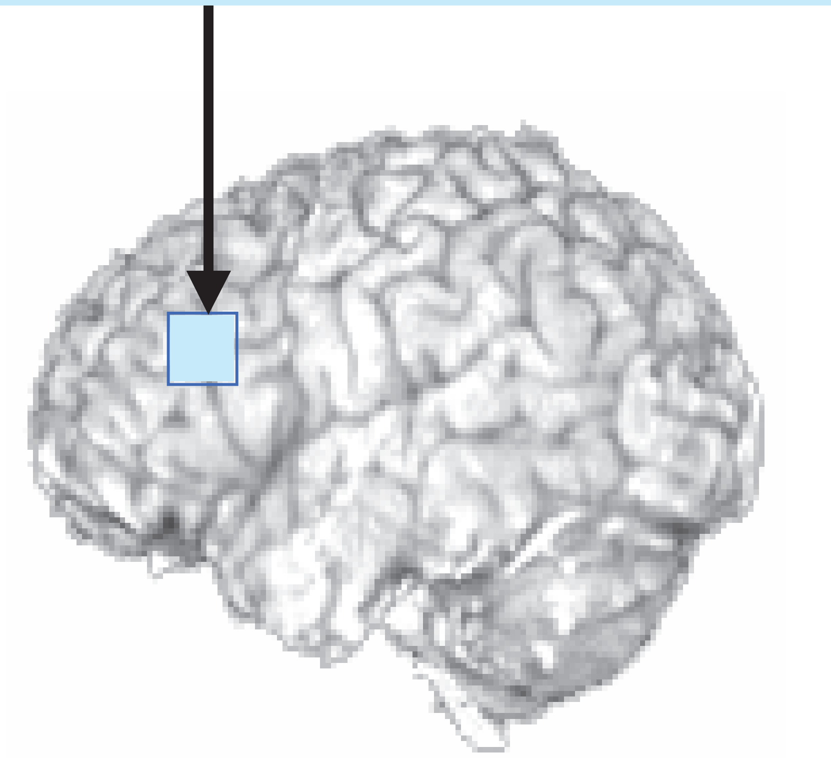
Motor/Manual: BA 3/4
(x = -37, y = -25, z = 47)

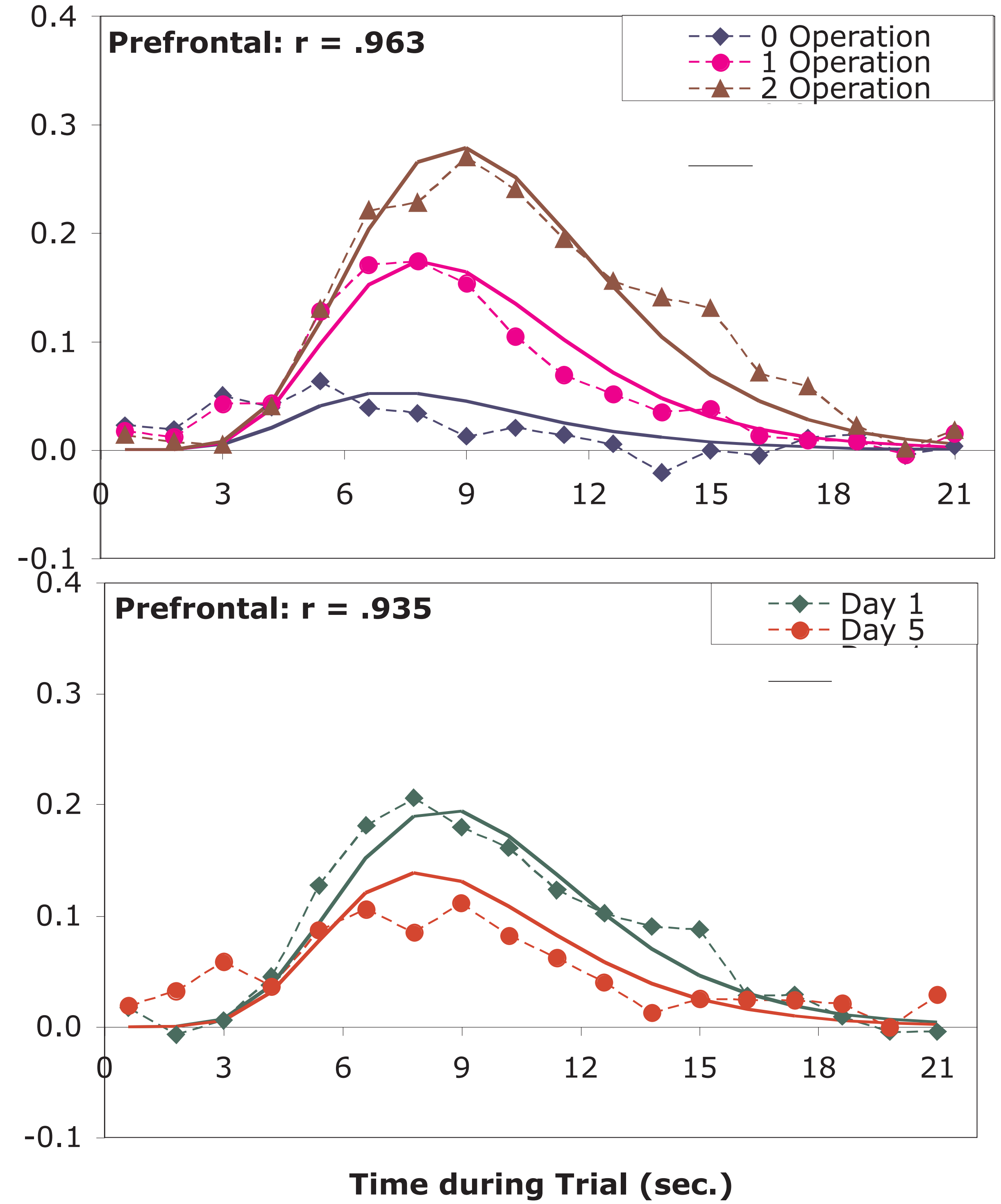




Results: Prefrontal/Retrieval

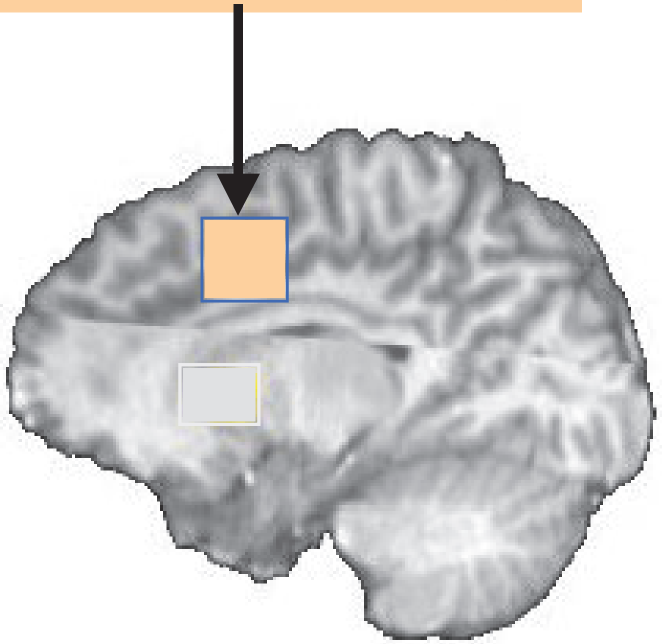
Prefrontal/Retrieval: BA 45/46
(x = -40, y = 21, z = 21)

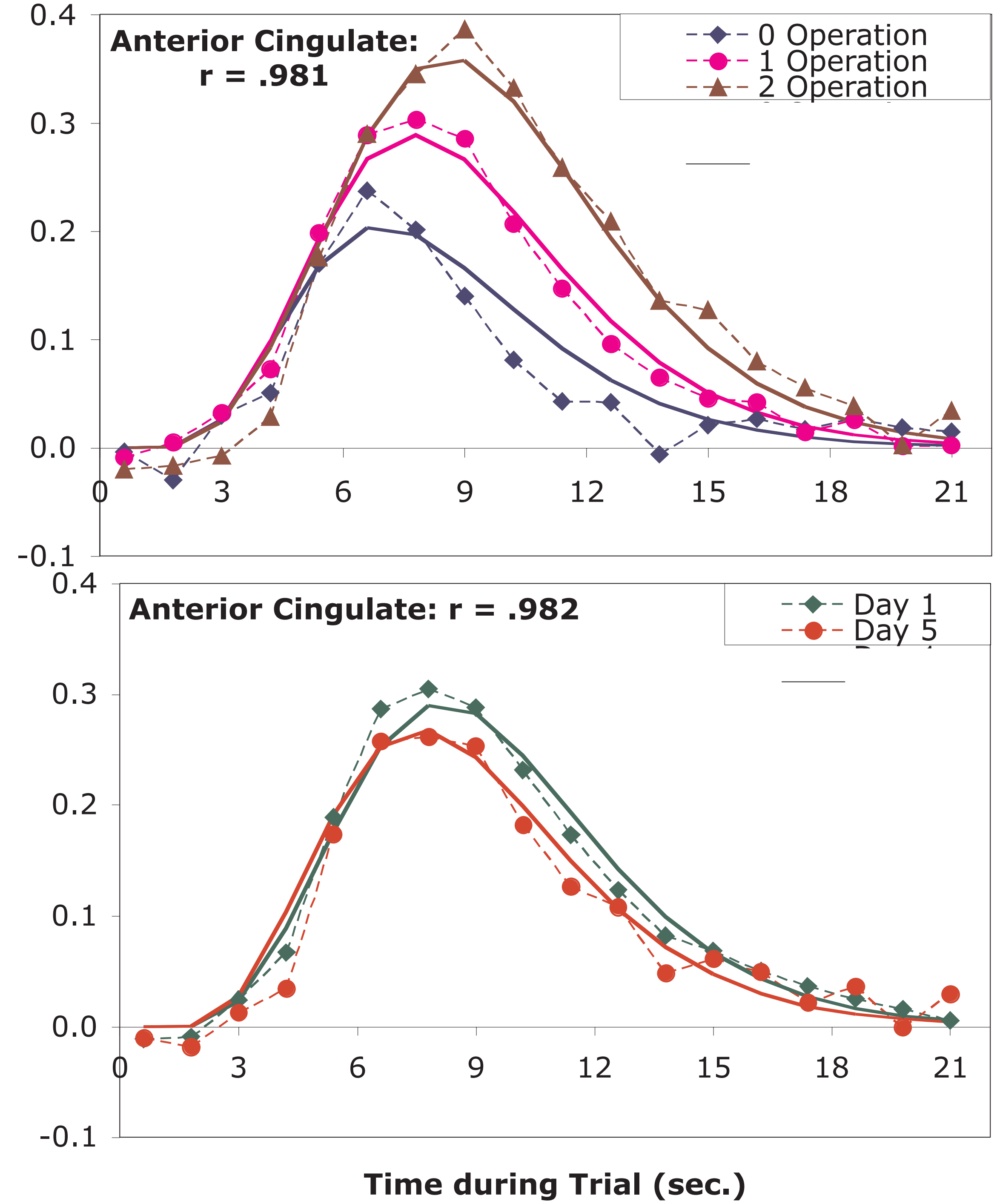




Results: AC/Goal

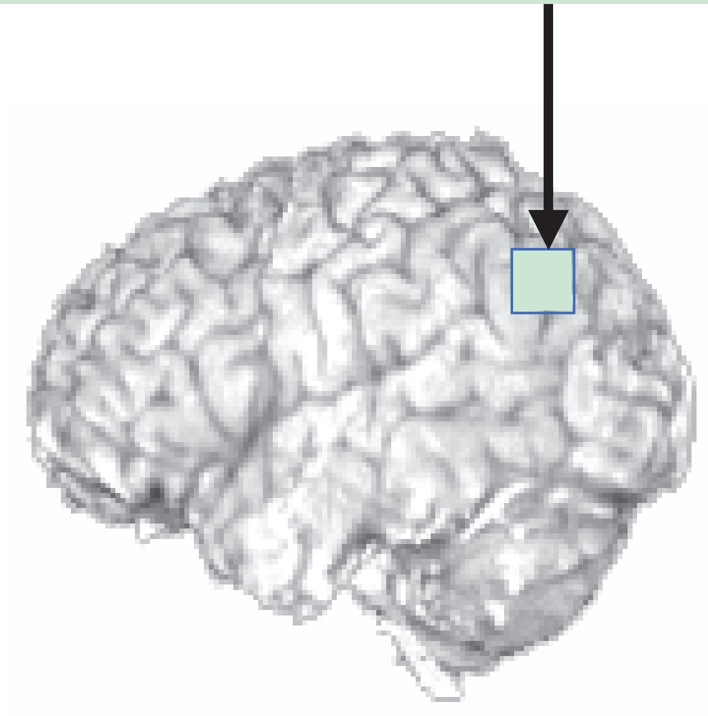
Ant Cing/Goal:BA 24/32
(x = -5, y = 10, z = 38)

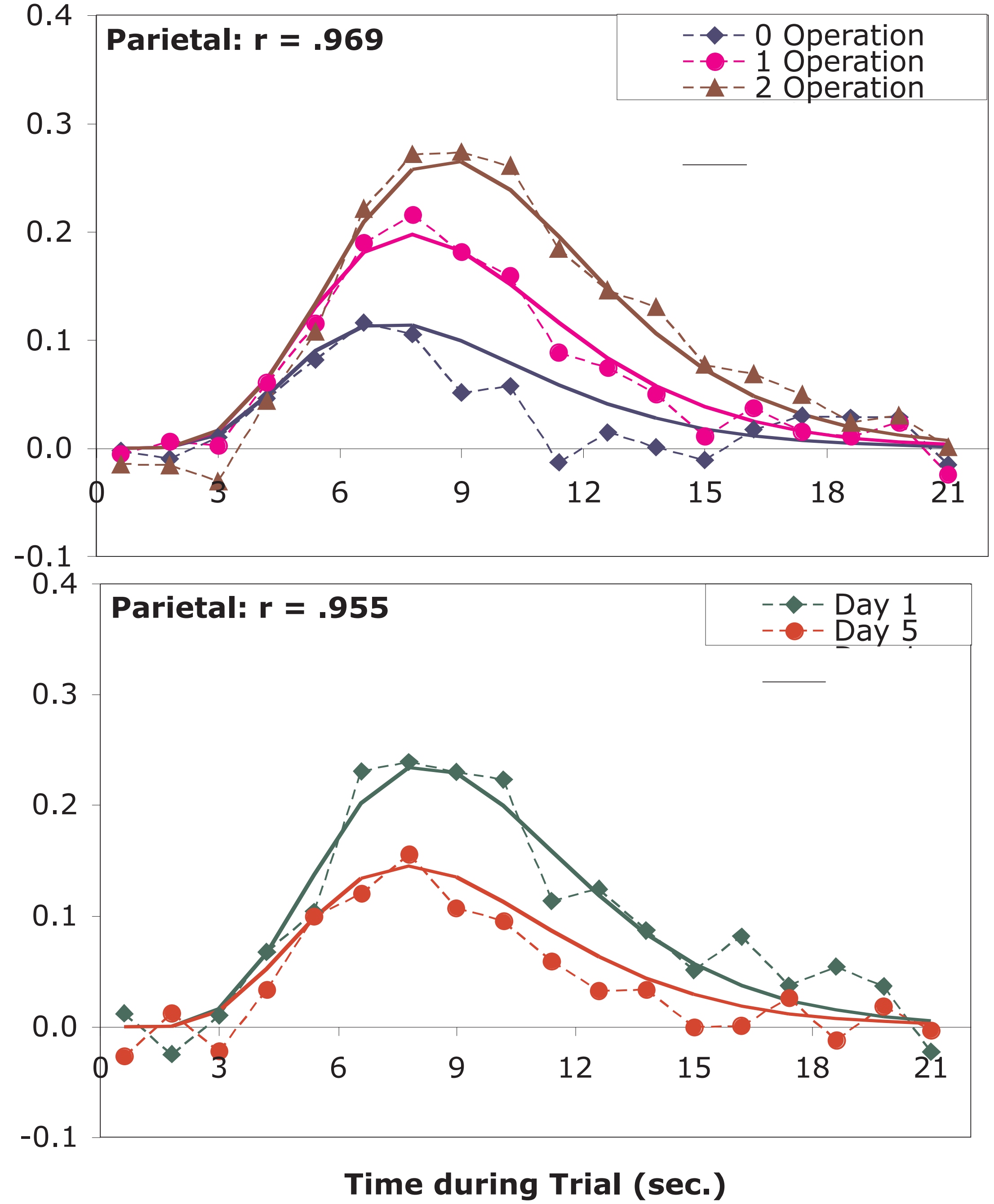




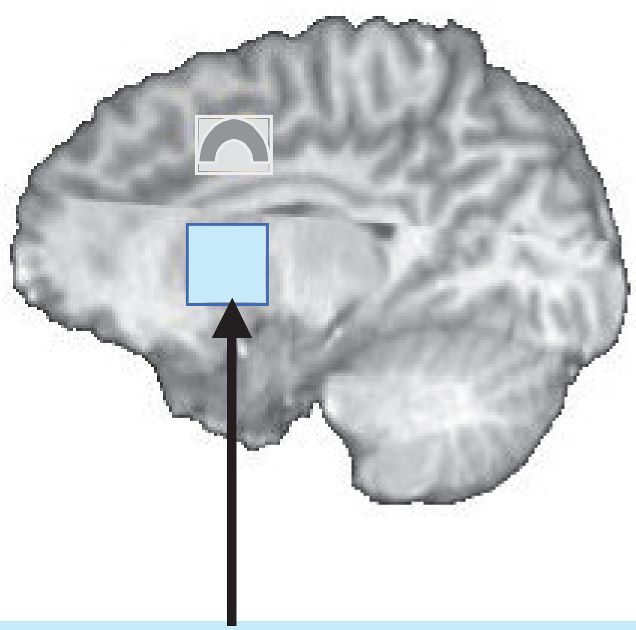
Results: Parietal/Imaginal

Parietal/Imaginal: BA 39/40
(x = -23, y = -64, z = 34)

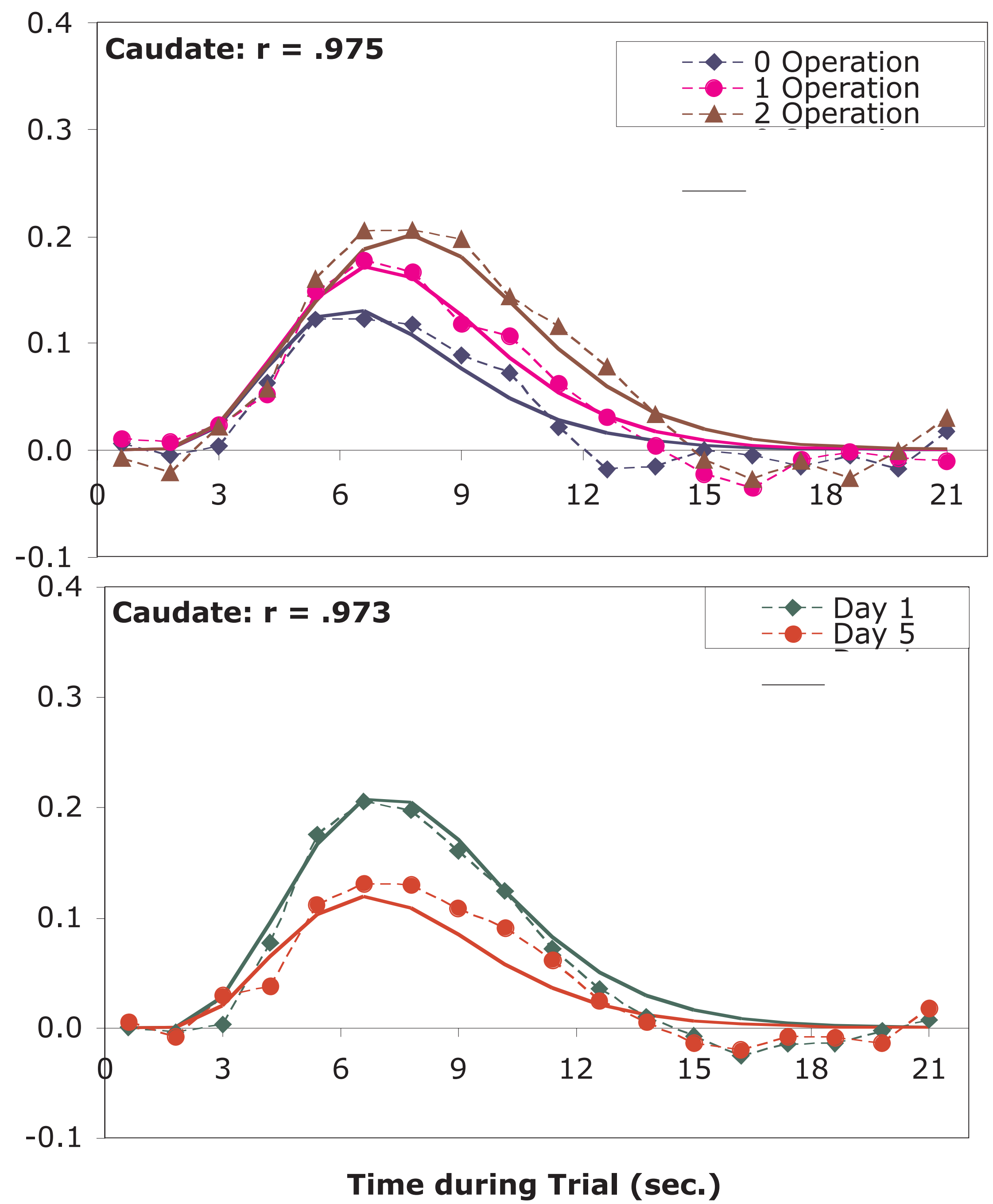




Results: Caudate/Procedural



Caudate/Procedural:
(x = -5, y = 9, z = 2)



Discussion & Conclusion

- Simple, yet complex...
- Algebra a "uniquely human" skill?
- Explanation by behavior production & convergence on multiple levels
- Unified theories vs. 20 questions (Newell, 1973)

Conclusion: Why bother?

- Cognitive science services:
  - Explore constraints (biological, developmental, cultural)
  - Directing questions
  - Methodologies
  - Applications, e.g., teaching & training

- A 'rational' basis of mathematics? mathematical mind as an adaptation to the structure of the world...

The End

Questions, comments, criticism...

CogWorks

http://www.cogsci.rpi.edu/cogworks/

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